MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Manifolds

Definition.

A set $M \subset \mathbb{R}^m$ is said to be a **differentiable** n**-manifold** provided that there exists a set of pairs $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ called *charts* satisfying the following:

1. Each $U_\alpha \subset M$ and $M = \bigcup_{\alpha \in I} U_\alpha$.
2. Each φ_α is a map from U_α to $\mathbb{R}^n$ for which $E_\alpha = \mathrm{Im}(\varphi_\alpha)$ is an open subset of $\mathbb{R}^n$. Furthermore, $\varphi_\alpha : U_\alpha \longrightarrow E_\alpha$ is a *homeomorphism*, that is, a bijection such that both φ_α and φ_α^{-1} are continuous.
3. The charts are compatible in the sense that if (U, φ) and (V, ψ) are two charts with $U \cap V \neq \emptyset$, then the map
$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$
is a *diffeomorphism*, that is, a homeomorphism such that both the function and its inverse are in the class C^∞, or in other words, are infinitely differentiable.

Remarks:

(i) Note that n and m are fixed throughout and that $1 \leq n \leq m$. (The case $n = 0$ would consist of a collection of isolated points at best.)

(ii) Note that both $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$.

(iii) Though it is not a formal part of the definition, the U_α are generally taken to be connected.

(iv) The topology of M is given by the *relative topology* from $\mathbb{R}^m$, where a set $S \subset M$ is open in M if there exists some open set $T \subset \mathbb{R}^m$ such that $S = T \cap M$. In this sense, the U_α are open as subsets of M.