Last Name: _____

First Name: _____

MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (in-class portion)
October 31, 2001

Directions: You have one hour for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

Problem	Points	Score
1	24	
2	24	
3	27	
4	25	
Total	100	

1. True or False

    T or **F**   Every Cauchy sequence of rational numbers converges to a rational number.

    T or **F**   If $L : V \longrightarrow W$ is a surjective linear map, then it is invertible.

    T or **F**   A linear transformation may have infinitely many eigenvalues.

    T or **F**   If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of linearly independent vectors, then any $\mathbf{v} \in V$ may be written as a linear combination of these vectors in a unique way.

    T or **F**   If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.

For the next three, consider the following:

Let K be the kernel of the linear transformation $L : V \longrightarrow V$.

    T or **F** K is a subspace of V.

    T or **F** K is an eigenspace of L.

    T or **F**   If K is finite-dimensional, then V/K is finite-dimensional.

2. Show that the linear transformation $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$A(x, y, z) = (x + y, y, 2z)$$

is not diagonalizable by following these steps:

(a) Define what it means for a linear transformation $L : V \longrightarrow V$ to be *diagonalizable*. (Hint: You may state either the definition or the theorem which gives an equivalent definition.)

(b) Determine all eigenvalues of A.

(c) Give bases for all the corresponding eigenspaces.

(d) Conclude that A is not diagonalizable.

3. Consider the linear transformation $B : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by

$$B(x, y, z, w) = (z - x, y - z, 0, x - y)$$

(a) Find a basis for $K = Ker(B)$, and show that it is a basis.

(b) Define the *quotient space* U/V, where V is a subspace of a vector space U.

(c) Find a basis for $\mathbb{R}^4/K$, and show that it is a basis.

4. If $d \in \mathbb{Z}$ is a non-square (that is, there is no integer c such that $c^2 = d$), then $\sqrt{d}$ is an irrational number, and we define

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \,|\, a, b \in \mathbb{Q}\},$$

where $\mathbb{Q}$ is the field of rational numbers. We also define addition and multiplication as follows:

$$(a + b\sqrt{d}) + (c + e\sqrt{d}) = (a + c) + (b + e)\sqrt{d}$$

$$(a + b\sqrt{d}) \cdot (c + e\sqrt{d}) = (ac + bed) + (ae + bc)\sqrt{d}$$

It may be checked that $\mathbb{Q}(\sqrt{d})$ is a vector space over the field $\mathbb{Q}$ by verifying the axioms. Note that scalar multiplication is given implicitly by the second formula above, where we identify the rational number (the scalar) $a \in \mathbb{Q}$ with the element $a + 0\sqrt{d} \in \mathbb{Q}(\sqrt{d})$. Moreover, according to the rules for addition and multiplication above, it may be verified that $\mathbb{Q}(\sqrt{d})$ is actually a field.

Finally, we define a concept closely related to that of linear map. If F_1 and F_2 are fields, then a *field homomorphism* between them is a map $\varphi : F_1 \longrightarrow F_2$ satisfying:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \forall x, y \in F_1$$

$$\varphi(xy) = \varphi(x)\varphi(y), \forall x, y \in F_1$$

(a) Find the dimension of $\mathbb{Q}(\sqrt{d})$ over $\mathbb{Q}$ (as a vector space) by finding a basis.

(b) Show that $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{-1})$ as vector spaces by constructing a bijective linear map between them.

(c) Show that $\mathbb{Q}(\sqrt{2})$ satisfies Axiom M4 (Multiplicative Inverses) for a field by finding the multiplicative inverse for the element $a + b\sqrt{2}$.

(d) Show that for any field homomorphism, $\varphi : F_1 \longrightarrow F_2$, we must have $\varphi(1) = 1$.

(e) Show that the map you constructed in part (b) is not a field isomorphism (bijective homomorphism).