

MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Solutions (in-class portion)
October 31, 2003

1. True or False

- (a) $\mathbb{Z}/4\mathbb{Z}$ is a field.

False. The ring $\mathbb{Z}/4\mathbb{Z}$ is not a field because it does not satisfy the multiplicative inverse axiom. In particular, $2 \cdot 2 = 0$.

- (b) $\mathbb{Z}/5\mathbb{Z}$ is a commutative ring with identity.

True. It is also a field, but all fields are commutative rings with identity.

- (c) In a vector space, additive inverses are unique.

True. We proved this.

- (d) The empty set is a vector space over any field.

False. One of the vector space axioms asserts the existence of an additive identity vector, $\mathbf{0}$.

- (e) Every non-trivial vector space contains infinitely many vectors.

False. Any finite-dimensional vector space over a finite field (such as $F = \mathbb{Z}/p\mathbb{Z}$) has only finitely-many vectors.

- (f) Every vector space (except the trivial vector space) has a basis.

True. This is a major theorem.

- (g) Any three non-zero vectors span \mathbb{R}^3 .

False. For example, consider $(1, 0, 0)$, $(2, 0, 0)$, and $(3, 0, 0)$.

- (h) The vector space \mathbb{R}^3 has a basis containing the vector $(1, 2, 3)$.

True. Any linearly independent set may be extended to create a basis.

- (i) **For p a prime, the vector space $V = (\mathbb{Z}/p\mathbb{Z})^2$ has $(p^2 - 1)(p^2 - p)$ (ordered) bases.**
True. There are $p^2 - 1$ possible non-zero vectors to be selected as the first vector in the basis and $p^2 - p$ vectors which are not scalar multiples of the first which could be selected as the second vector in the basis.
- (j) **If $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then $\dim(V) = n$.**
False. The vectors might not be linearly independent.
- (k) **If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.**
False. Consider $V = \mathbb{R}^3$ and the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.
- (l) **If $V \cong W$, then there is a bijective linear map $L : V \rightarrow W$.**
True. This is the definition of isomorphism.
- (m) **Isomorphism of vector spaces is an equivalence relation.**
True. This follows from the properties of bijections.
- (n) **If V is a subspace of U and $\dim(V) < \infty$, then $\dim(U/V) < \infty$.**
False. Consider $U = C^\infty(\mathbb{R})$ and $V = P_0(\mathbb{R}) = \{\text{constant functions}\}$.

For the remainder of the True/False questions, let $L : U \rightarrow V$ be a linear map.

- (a) **If $\text{Ker}(L) = \{\mathbf{0}\}$, then L is injective.**
True. We proved this.
- (b) **If $\dim(U) = \dim(V)$, then L is bijective.**
False. Consider the case when L is the zero map.
- (c) **$\text{Ker}(L)$ is a subspace of U .**
True. We proved this.
- (d) **If $\dim(U) < \infty$, then $\dim(\text{Im}(L)) < \infty$.**
True. Choose a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ for U . Then $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ will at least span $\text{Im}(L)$, so we can choose a basis for $\text{Im}(L)$ from this set.

- (e) **If $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis for U , then $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is a basis for V .**
False. Consider the case when L is not injective.
- (f) **If $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a linearly independent set, then $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is also a linearly independent set.**
False. Consider the case when L is not injective.

2. State the Principle of Mathematical Induction.

Two possibilities:

1. (Written in the language of Peano's Postulates where P is the set being characterized and $S : P \rightarrow P$ is the successor function.)

Suppose $P' \subset P$ satisfies the following properties:

- $1 \in P'$
- If $n \in P'$, then $S(n) \in P'$.

Then $P' = P$.

2. (Written in the language of the natural numbers \mathbb{N} .)

Suppose there is a subset X of natural numbers with the following properties:

- $1 \in X$
- If $n \in X$, then $n + 1 \in X$.

Then $X = \mathbb{N}$.

3. Let $L : U \rightarrow V$ be a linear map between two finite-dimensional vector spaces, and suppose that $\dim(U) > \dim(V)$.

- (a) **Define (in terms of the elements of U and V) what it would mean for L to be *surjective*.**

The map L is *surjective* if, given any $\mathbf{v} \in V$, there is some $\mathbf{u} \in U$ such that $L(\mathbf{u}) = \mathbf{v}$.

(b) Multiple choice (circle one):

- i. L must be surjective.
- ii. L could be, but is not necessarily, surjective.
- iii. L cannot be surjective.

The answer is ii. It is a useful exercise to come up with examples of both possibilities.

(c) Use the Rank-Nullity Theorem to prove that L cannot be injective.

The Rank-Nullity Theorem states that, in the context of the linear map L above:

$$\dim(\text{Im}(L)) + \dim(\text{Ker}(L)) = \dim(U).$$

We will show that L is not injective by showing that it has a non-trivial kernel. That is, we will show that $\dim(\text{Ker}(L)) > 0$. Now, we know that $\text{Im}(L)$ is a subspace of V and hence

$$\dim(\text{Im}(L)) \leq \dim(V).$$

Combining this with Rank-Nullity, we get the following inequality:

$$\dim(\text{Ker}(L)) \geq \dim(U) - \dim(V).$$

Finally, our hypothesis says that $\dim(V) < \dim(U)$, so the right-hand side of the preceding inequality is strictly positive, which is what we were trying to show.

4. For any $n \in \mathbb{N}$, recall that we define

$$P_n(\mathbb{R}) = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}, \forall i\}$$

to be the vector space of polynomials of degree less than or equal to n with real coefficients.

(a) Construct a basis for $P_3(\mathbb{R})$ containing $p(x) = 1 - x + x^2 - x^3$ as one of its vectors.

(You don't need to show it to be a basis in this part.)

Two examples: $\mathfrak{B}_1 = \{1, x, x^2, p(x)\}$ and $\mathfrak{B}_2 = \{p(x), x, x^2, x^3\}$

(b) Show that your basis from part (a) does in fact span $P_3(\mathbb{R})$.

We do this for example \mathfrak{B}_1 from part (a).

Suppose $q(x) = a + bx + cx^2 + dx^3$ is any polynomial in $P_3(\mathbb{R})$. Then we must be able to write it as a linear combination of the elements of \mathfrak{B}_1 . In other words, we need real numbers α , β , γ , and δ such that

$$q(x) = \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2 + \delta \cdot p(x).$$

This single equation can be turned into a set of four equations in four unknowns (namely, α , β , γ , and δ), which can then be solved simultaneously. The easier method (because we chose a simple basis, \mathfrak{B}_1) is to observe that the only way to match coefficients on the x^3 term is if $\delta = -d$. It follows quickly that $\alpha = a + d$, $\beta = b - d$, and $\gamma = c + d$. We can then write q as:

$$q(x) = (a + d) \cdot 1 + (b - d) \cdot x + (c + d) \cdot x^2 + (-d) \cdot p(x).$$

(c) Give an example of a non-zero linear map

$$L : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R}).$$

Three examples (the first two of which were common responses, the third of which is the easiest I can think of):

Let $D : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be the standard differentiation operator, that is, $D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$.

Let $T : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be the truncation operator given by $T(a + bx + cx^2 + dx^3) = a + bx + cx^2$.

Let $A : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ simply be $A(a + bx + cx^2 + dx^3) = a$.

(d) Show that your map L from part (c) is in fact linear.

We show this for the map D from part (c). Let $p_1(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and let $p_2(x) = a_2 + b_2x + c_2x^2 + d_2x^3$. Then:

$$\begin{aligned} D(p_1(x) + p_2(x)) &= D((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3) \\ &= (b_1 + b_2) + 2(c_1 + c_2)x + 3(d_1 + d_2)x^2 \\ &= (b_1 + c_1x + d_1x^2) + (b_2 + c_2x + d_2x^2) \\ &= D(p_1(x)) + D(p_2(x)), \end{aligned}$$

and

$$\begin{aligned}D(\alpha \cdot p_1(x)) &= D(\alpha \cdot (a_1 + b_1x + c_1x^2 + d_1x^3)) \\&= (\alpha a_1) + (\alpha b_1)x + (\alpha c_1)x^2 + (\alpha d_1)x^3 \\&= \alpha b_1 + 2\alpha c_1x + 3\alpha d_1x^2 \\&= \alpha \cdot (b_1 + c_1x + d_1x^2) \\&= \alpha \cdot D(p_1(x)).\end{aligned}$$

5. Let $\ell = \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R}, \forall i \in \mathbb{N}\}$ be the vector space of all sequences of real numbers, and consider the shift operator $S : \ell \rightarrow \ell$ given by:

$$S(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots).$$

Let S^2 and S^3 be the composition of S with itself 2 and 3 times, respectively, and let $T = S^3 - S^2 - S$.

- (a) For a vector $(a_1, a_2, a_3, \dots) \in \ell$, write out $T(a_1, a_2, a_3, \dots)$ explicitly, including at least the first four terms of the resulting sequence.

$$T(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots) = (a_4 - a_3 - a_2, a_5 - a_4 - a_3, a_6 - a_5 - a_4, a_7 - a_6 - a_5, \dots)$$

- (b) Find a basis for $\text{Ker}(T)$.
(You don't need to show it to be a basis in this part.)

If $T(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots) = (0, 0, 0, \dots)$, then we get the following system of equations:

$$\begin{aligned}a_4 - a_3 - a_2 &= 0 \\a_5 - a_4 - a_3 &= 0 \\a_6 - a_5 - a_4 &= 0 \\a_7 - a_6 - a_5 &= 0 \\&\vdots\end{aligned}$$

These may be re-written as:

$$\begin{aligned}a_4 &= a_2 + a_3 \\a_5 &= a_3 + a_4 \\a_6 &= a_4 + a_5 \\a_7 &= a_5 + a_6 \\&\vdots\end{aligned}$$

In order to solve these equations, we see that we may choose a_1 , a_2 , and a_3 however we like, and then the equations will determine the rest of the a_i for $i \geq 4$ in terms of our choices.

For simplicity, we allow each of a_1 , a_2 , and a_3 in turn to be 1 while keeping the others 0. These choices produce the following vectors:

$$\mathbf{v}_1 = (1, 0, 0, 0, 0, 0, 0, \dots)$$

$$\mathbf{v}_2 = (0, 1, 0, 1, 1, 2, 3, \dots)$$

$$\mathbf{v}_3 = (0, 0, 1, 1, 2, 3, 5, \dots)$$

where, for \mathbf{v}_2 and \mathbf{v}_3 , the succeeding terms are always determined by the equation $a_k = a_{k-1} + a_{k-2}$ for $k \geq 7$. (Note that there were no restrictions whatsoever on the choice of a_1 and that a_1 does not influence any of the later terms.)

(c) Show that your basis from part (b) is in fact linearly independent.

Suppose there were some linear combination of \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 giving the zero vector. In other words, suppose there were real numbers a , b , and c such that

$$a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}.$$

Writing this out in long form gives

$$(a, b, c, b + c, b + 2c, 2b + 3c, 3b + 5c, \dots) = (0, 0, 0, 0, 0, 0, 0, \dots)$$

and clearly, by comparing the first three terms, we see that we must have $a = 0$, $b = 0$, and $c = 0$, and thus the three vectors are linearly independent.