MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Exam (take-home portion)
Due: 11 A.M. on November 7, 2003

Directions: For this exam, you will work in teams of four students, which should be confirmed with the instructor before beginning the exam. You have until 11 A. M. on Friday, November 7, to complete this exam, at which time it should be submitted in class. You may use your own class notes (or those of your teammates), your own homework assignments (or those of your teammates), and the course textbooks (and supplemental bibliography) as your only aids. You may not use any internet resources except for the course website and the posted homework and exam solutions.

You may discuss the exam only with your teammates, and all questions should be directed only to the instructor. Obviously, the course will be continuing, and you should be going to section, so you will be able to speak with the Math 23a CA's, but you should not put them in an awkward position by asking questions directly related to the take-home, and they will be instructed to report anyone attempting to gain unfair advantage.

Each team should submit one solution set, with each team member hand-writing a solution to one of the problems. There is partial credit, but only for intelligible work. One point per problem will be awarded for *neatness only*, and one point will be awarded for *mathematical style only*, including brevity. Make sure your name and your teammates' names are prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take from a textbook. Otherwise, all work should be that of you and your team.

1. Let U, V, and W be vector spaces (not necessarily finite-dimensional) over the same field, and suppose we have the following four linear maps:

$$A : \{\mathbf{0}\} \longrightarrow U$$

$$S : U \longrightarrow V$$

$$T : V \longrightarrow W$$

$$B : W \longrightarrow \{\mathbf{0}\}$$

with the property that the kernel of any of them is equal to the image of the previous one. In other words, $Ker(S) = Im(A)$, $Ker(T) = Im(S)$, and $Ker(B) = Im(T)$.

Such a collection of vector spaces and linear maps is known as a *(short) exact sequence*, which is often denoted by:

$$\{\mathbf{0}\} \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow \{\mathbf{0}\}$$

(a) Show that S is injective.

(b) Show that T is surjective.

(c1) Show that $U \cong S(U)$.

(c2) Show that $W \cong V/U$.
(Note that this is an abuse of notation since U is not, strictly speaking, a subspace of V. We are using part (c1) to identify U with its image $S(U)$.)

(d) Consider the case when $V = C^\infty(\mathbb{R})$, the vector space of all infinitely differentiable real-valued functions, and the map T is differentiation, that is $T(f) = f'$. What choice of a vector space U and a map S make $\{\mathbf{0}\} \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow \{\mathbf{0}\}$ into an exact sequence? (Bonus: What subspace of $C^\infty(\mathbb{R})$ is W in this case?)

2. Consider the field $F = \mathbb{Z}/2\mathbb{Z}$ with its elements identified as 0 and 1. (Properly speaking, these are representatives of equivalence classes, but we will allow the simpler notation.)

   Now define $F[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \in \mathbb{N}, a_i \in F, \forall i\}$ to be the vector space of all polynomials with coefficients in $F = \mathbb{Z}/2\mathbb{Z}$, where addition and scalar multiplication are defined as usual. Note, however, that we also know how to multiply two polynomials, and that with this notion of multiplication, $F[x]$ actually satisfies the axioms for a ring.

   Let $p(x) = x^3 + x + 1$ be a fixed vector (polynomial) in $F[x]$, and define $I = \{a(x)p(x) \mid a(x) \in F[x]\}$ to be the subspace of $F[x]$ consisting of all multiples (polynomial multiples, not just scalar multiples) of this single vector.

   (a) Show that I is a subspace of vector space $F[x]$. (In fact, it is a *subring*, but we are only concerned with vector space properties in this part of the question.)

   (b) Define the quotient space $F[x]/I$ in terms of the data above, and find a minimal complete set of coset representatives. (Note that this is not the same as finding a basis. Since F is finite, it is possible to list all the elements of $F[x]/I$. Hint: You might consider long division of polynomials to help you classify the cosets.)

   (c) We have already seen in general that the quotient space has the structure of a vector space (so that we already have addition and scalar multiplication). Show that the natural definition of multiplication is well-defined for elements of the quotient space $F[x]/I$.

   (d) With the multiplication from part (c) and the representatives from part (b), show that $F[x]/I$ satisfies Axiom M3 and M4 (multiplicative identity and inverses) for a field by explicitly naming the identity and all multiplicative inverses. (This shows that $F[x]/I$ is a field because Axioms M1, M2, and D are inherited from the stucture of $F[x]$.)

3. We define a linear map $P : V \longrightarrow V$ to be a *projection* provided that $P^2 = P$. For example, $P = 0$ (the zero map) and $P = I$ (the identity map) are projections since $0^2 = 0$ and $I^2 = I$, but these are trivial projections.

   In the following, consider a projection P.

   (a) Show that 0 and 1 are the only possible eigenvalues of P.

   (b) If V_λ represents the eigenspace for the eigenvalue λ under the map P, show that $V \cong V_0 \oplus V_1$. (Note that this shows that a projection is diagonalizable!)

   (c) For the case $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $P(x, y) = (ax + by, cx + dy)$, find all possible projections. That is, find conditions on a, b, c, and d such that P is a projection if and only if your conditions are met.

   (d) For a non-trivial projection from part (c), describe geometrically what is happening to the vector $(x, y) \in \mathbb{R}^2$ when you apply P. Please use pictures! (Even though we have not defined them formally yet, you may use the terms *parallel*, *perpendicular*, etc. from Euclidean geometry.)

4. Let V and W be vector spaces over the field F, and let $\mathcal{L}(V, W)$ denote the set of all linear maps from a V to W. Define addition and scalar multiplication in $\mathcal{L}(V, W)$ as follows:

$$\begin{aligned} (T_1 + T_2)(\mathbf{v}) &= T_1(\mathbf{v}) + T_2(\mathbf{v}) \\ (c \cdot T)(\mathbf{v}) &= c \cdot T(\mathbf{v}) \end{aligned}$$

   (a) Prove that $\mathcal{L}(V, W)$ is a vector space with addition and scalar multiplication as above.

   (b) Consider the case when V and W are finite-dimensional, say $\dim(V) = m$ and $\dim(W) = n$. Find a basis for $\mathcal{L}(V, W)$, and compute its dimension.

   (c) In the special case where $W = F$, we call $\mathcal{L}(V, F)$ the *dual space* of V, which we denote V^*. Since V^* is itself a vector space over F, we may then find *its* dual, $(V^*)^* = \mathcal{L}(V^*, F)$.
   Show that $(V^*)^* \cong V$ by exhibiting a natural bijective linear map between them.