**Last Name:** ______________________________

**First Name:** ______________________________

MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (in-class portion)
March 19, 2003

**Directions**: You have one hour for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

| Problem | Points | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 6 | |
| 5 | 6 | |
| Total | 54 | |

1. **True or False**

- **T** or **F** Every bounded infinite set in $\mathbb{R}^n$ has an accumulation point.

- **T** or **F** Let $A \subset \mathbb{R}^n$. If $f : A \longrightarrow \mathbb{R}$ is continuous and $f$ attains its maximum on $A$, then $A$ is compact.

- **T** or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is connected, then $f(S) \subset \mathbb{R}^m$ is connected.

- **T** or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, then all of its directional derivatives exist at $\mathbf{a} \in \mathbb{R}^n$.

- **T** or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a}$, and $\mathbf{h}$ is some direction vector, then $D_{\mathbf{h}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{h}$.

- **T** or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, then $[D_i D_j f](\mathbf{a}) = [D_j D_i f](\mathbf{a})$, for all $i$ and $j$.

- **T** or **F** On the set of points $S = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$, where $f(x, y) = x^2 - y^2 - 1$, there is a neighborhood of the point $(1, 0)$ on which we may write $y = h(x)$ with $f(x, h(x)) = 0$.

- **T** or **F** Let $A \in M_n(\mathbb{R})$. Then $A$ is symmetric if and only if $A$ has an orthonormal eigenbasis.

- **T** or **F** The quadratic form $q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $q(x, y) = x^2 + 4xy + y^2$ is positive-definite.

2. **Compactness.** (12 points, 3/3/6)

   (a) Define what it means for a set $A \subset \mathbb{R}^n$ to be compact.

   (b) State the Heine-Borel Theorem.

   (c) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$ is compact. Show that $f(A) \subset \mathbb{R}^m$ is compact.

3. **Differentiability.** (12 points, 3/3/6)

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Define what it means for $f$ to be *differentiable* at $\mathbf{a} \in \mathbb{R}^n$.

(b) Give an example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ that is continuous at all $\mathbf{x} \in \mathbb{R}^n$, but which is not differentiable at some point $\mathbf{a} \in \mathbb{R}^n$.

(You should specify $\mathbf{a}$, but you need not prove the continuity or the lack of differentiability of $f$, as long as the example is correct.)

(c) Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is bounded (in other words, $\exists M > 0$ such that $|f(\mathbf{x})| \leq M, \forall \mathbf{x} \in \mathbb{R}^n$), and define $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$g(x, y) = x \cdot y \cdot f(x, y).$$

Show that $g$ is differentiable at $(0, 0)$.

4. **Inverse Function Theorem.** (6 points, 3 each)

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by:

$$f(x, y) = (e^{xy}, e^{x+y})$$

(a) Find the Jacobian, $Jf$.

(b) Find all points $(x, y) \in \mathbb{R}^2$ at which the Inverse Function Theorem does not apply to $f$.

5. **Optimization of Functions.** (6 points)

Use critical point classification and the method of Lagrange multipliers to find the point(s) on the closed unit sphere

$$D^3 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$$

(which includes its interior) where the function

$$f(x, y, z) = x^3 + y^3 + z^3$$

attains its maximum and minimum.