Problem Set 1, Part B – Solutions

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2(B) Verify that multiplication is well defined for integers, as defined by equivalence classes.

Solution To check whether multiplication is well-defined, we have to show that given three equivalence classes $[(a_1, b_1)], [(a_2, b_2)]$ and $[(a_3, b_3)]$ such that $(a_1, b_1) \sim (a_2, b_2)$ then:

$$[(a_1, b_1)] \cdot [(a_3, b_3)] = [(a_2, b_2)] \cdot [(a_3, b_3)]$$

In other words, what we have to show is that by changing the representative of a class, we don't change the final outcome.

Since $(a_1, b_1) \sim (a_2, b_2) \Rightarrow a_1 + b_2 = a_2 + b_1$. Multiply this equation first by a_3 and then by b_3, both to the right, and obtain the following two equations:

$$\begin{aligned} a_1a_3 + b_2a_3 &= a_2a_3 + b_1a_3 && (1) \\ a_1b_3 + b_2b_3 &= a_2b_3 + b_1b_3 && \Rightarrow \text{(reverse)} \\ a_2b_3 + b_1b_3 &= a_1b_3 + b_2b_3 && (2) \end{aligned}$$

Add equations 1 and 2 (remember we are not allowed to subtract yet; that is why we are still working with equivalence classes):

$$a_1a_3 + b_2a_3 + a_2b_3 + b_1b_3 = a_2a_3 + b_1a_3 + a_1b_3 + b_2b_3$$

Now use commutativity and associativity for the natural numbers:

$$\begin{aligned} (a_1a_3 + b_1b_3) + (a_2b_3 + b_2a_3) &= (a_1b_3 + a_3b_1) + (a_2a_3 + b_2b_3) \\ (a_1b_3 + a_3b_1, a_1a_3 + b_1b_3) &\sim (a_2b_3 + b_2a_3, a_2a_3 + b_2b_3) \\ [(a_1, b_1)] \cdot [(a_3, b_3)] &= [(a_2, b_2)] \cdot [(a_3, b_3)] \end{aligned}$$

Similarly, one can show that:

$$[(a_3, b_3)] \cdot [(a_1, b_1)] = [(a_3, b_3)] \cdot [(a_2, b_2)]$$

In conclusion, multiplication is well-defined.