Math 23a Solution Set #1, Part C

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Problem 5

(a) *Make a good definition of the rational numbers as equivalence classes of pairs of integers.*

Solution: Define $\mathbb{Q} = \{(a,b) \in \mathbb{Z}^2 \mid b > 0\}/\sim$, where $\sim$ is the equivalence relation given by

$$(a,b) \sim (c,d) \qquad \text{iff} \qquad ad = cb.$$

One can easily check $\sim$ is reflexive, symmetric, and transitive.

(b) *For two rational numbers a and b (as you defined them), define addition ($a+b$), multiplication ($a \cdot b$), and less than ($a < b$).*

Solution: Given two rational numbers $[(a,b)], [(c,d)] \in \mathbb{Q}$, we define the following operations:

$$[(a,b)] + [(c,d)] \stackrel{\text{def}}{=} [(ad+cb, bd)]$$

$$[(a,b)] + [(c,d)] \stackrel{\text{def}}{=} [(ac, bd)]$$

$$[(a,b)] < [(c,d)] \quad \text{iff} \quad ad < cb.$$

Strictly speaking, you should check that these operations are well defined. For example, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then you should verify that $[(a,b)]+[(c,d)] = [(a',b')] + [(c',d')]$.

(c) *Prove the existence of additive inverses.*

Solution: Let $0_{\mathbb{Q}} = [(0,1)]$. It is easy to check that $0_{\mathbb{Q}}$ is an additive identity for $\mathbb{Q}$:

$$[(a,b)] + [(0,1)] = [(a \cdot 1 + 0 \cdot b, b \cdot 1)] = [(a,b)], \qquad \forall [(a,b)] \in \mathbb{Q}.$$

To prove the existence of additive inverses, we must show that for any $[(a,b)] \in \mathbb{Q}$, there exists $(-[(a,b)]) \in \mathbb{Q}$ such that $[(a,b)] + (-[(a,b)]) = [(0,1)]$. Indeed, let $(-[(a,b)]) \stackrel{\text{def}}{=} [(-a,b)]$, then

$$\begin{aligned} [(a,b)] + (-[(a,b)]) &= [(a,b)] + [(-a,b)] \\ &= [(ab + (-a)b, b^2)] \\ &= [(0, b^2)] \\ &= [(0,1)] \qquad \text{(since } (0,1) \sim (0,b^2) \text{ for any } b \in \mathbb{Z}). \end{aligned}$$

(d) *Prove that the positive rational numbers are closed under addition.*

Solution: We will say that a rational number $[(a,b)]$ is *positive* if $0_{\mathbb{Q}} < [(a,b)]$, that is, if $a > 0$. So let $[(a,b)], [(c,d)]$ be two positive rational numbers. Their sum is $[(ad+bc, bd)]$. Since $a,b,c,d > 0$, we know that $ad+bc > 0$, hence $0_{\mathbb{Q}} < [(ad+bc, bd]$. This shows that the positive rationals are closed under addition.

Remarks

- In general, **people were writing way too much**! Please keep your answers **concise**. A lot of people wrote down line after line of equations without explaining what they were doing. It is important to include details in your proof, but it is more important to let the grader know *what* you are writing about, and *why*. Your proofs should be written in clear English sentences. If you want to get a better idea of what a proof should look like, please take a look at Joe Rabinoff's "A Note on Proofs" on the website.

- Many people defined the rational numbers as equivalence classes of ordered pairs of integers (a,b) with the restriction $a \neq 0$ or the restriction $b \neq 0$. Either of these is fine. The only problem is that your definition of "less than" in part (b) will be more complicated (allowing $b < 0$), and this makes part (d) much messier.

- Very few people checked that their equivalence relation in (a) was actually an equivalence relation, and even fewer checked that the operations in (b) were well-defined. These are minor errors, so I didn't take off points for those.

- Several people defined the rationals simply as "ordered pairs of integers". You need to define the rationals as *equivalence classes* of ordered pairs. Therefore, when you're writing a rational number, it should be written as an equivalence class $[(a,b)]$ instead of simply an ordered pair (a,b).

- Some people "defined" an additive identity for $\mathbb{Q}$. You can't just define it; you need to *check that an additive identity exists*. In the solution above, I showed that $[(0,1)]$ was an additive identity by noting $[(a,b)] + [(0,1)] = [(a,b)]$ for all $[(a,b)] \in \mathbb{Q}$.

- Finally, for part (d), you need to use the definition of "less than" you stated in part (b). Many people lost points because they tweaked their definition without saying why. Some people wrote something like "if $[(a,b)] > 0$ and $[(c,d)] > 0$, then $[(a,b)] + [(c,d)] > 0 + 0 = 0$", but this is exactly what you're trying to prove!