Math 23a, Fall 2003

Problem Set 3, Part B
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Problem 3: *Let S be a set of some finite cardinality n. Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined are for functions of one real variable.*

Proof. Without loss of generality we may assume that $S = \{1, 2, \ldots, n\}$. We claim that a basis for V are the functions $f_1, f_2, \ldots, f_n$, where f_i $(1 \le i \le n)$ is defined by $f_i(j) = 0$ if $i \ne j$ and $f_i(j) = 1$ if $i = j$ for $1 \le j \le n$.

To show that $f_1, f_2, \ldots, f_n$ form a basis for V we need to show that they are linearly independent and that they span. Linear independence follows from the fact that the value of $\lambda_1 f_1 + \ldots \lambda_n f_n$ at j is λ_j, so $\lambda_1 f_1 + \ldots + \lambda_n f_n = 0$ if and only if $\lambda_1 = \ldots = \lambda_n = 0$. To show that $f_1, \ldots, f_n$ span let us take arbitrary $f \in V$. Then one directly checks that $f = \lambda_1 f_1 + \ldots + \lambda_n f_n$, where $\lambda_i = f(i)$ and we are done with this problem.

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