Solution for HW3, part C
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Problem 4

Let $V = \{(a_0, a_1, a_2, \ldots)|a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let W be the subspce of V consisting of all *arithmetic* sequences. Find a basis for W, and determine the dimension of W. *(A sequence is arithmetic if there is some constant c such that $a_{n+1} = a_n + c$ for all $n \geq 0$.)*

Solution

Everyone found a basis and showed their basis spanned W without trouble, but some neglected to demonstrate linear independence. Several people also forgot to note the dimension of the space. Read the problem carefully! This is a silly reason to lose a point.

Some possible bases I saw in the homeworks are:

$((\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \ldots), (\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots))$, or

$((\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots), (\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots))$, for example. The dimension, or cardinality of any basis set, is of course 2.

A slightly more general approach to the problem might be the following. Note that the map(function) $f : W \to \mathbb{R}^2$ which takes $(a_0, a_1, a_2, \ldots) \mapsto (a_0, a_1 - a_0)$, is an isomorphism of vector spaces. Or, stated another way,

$$f((a_0, a_0 + c, a_0 + 2c, a_0 + 3c, \ldots)) = (a_0, c) \in \mathbb{R}^2.$$

You can check that the map satisfies the properties of an isomorphism of vector spaces, i.e. that it is *linear* ($f(v + w) = f(v) + f(w)$ and $f(a \cdot v) = a \cdot f(v), a \in \mathbb{R}, v, w \in W$) and that it is *one-to-one* and *onto*. This observation is just a more formal way of saying that an arithmetic series is is determined uniquely by its initial term a_0 and the constant c.

Once we've established this isomorphism, life is easy, because now everything we know about $\mathbb{R}^2$ now applies to W (remembering that we can think of W and $\mathbb{R}^2$ as the *same* vector space–the map f just assigns a different name to each element of W without changing the vector-space structure).

It follows that $dim(W) = 2$ and that any basis for $\mathbb{R}^2$ gives a basis for W. Notice that $f((1, 1, \ldots)) = (1, 0)$ and $f((0, 1, 2, \ldots)) = (0, 1)$, so the first basis above corresponds to the standard basis of $\mathbb{R}^2$. The second basis above corresponds to the basis $((1, 1), (0, 1))$.