

Math 23a, Fall 2003

Problem Set 4, Part C
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Problem 6: Consider the linear differential operator $D : C^\infty \rightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number.

(a) Find $\ker(D)$.

(b) Show that D is surjective.

Proof. (a) $f \in \ker(D) \Leftrightarrow D(f) = 0 \Leftrightarrow f' + af = 0$. Set $y = f(x)$. Then the kernel is the solutions of $dy/dx + ay = 0$. Separating the variables and integrating gives us $\ln|y| = -ax + C$, which gives us the general solution $y = ce^{-ax}$, where $c \geq 0$ ($c = e^C$).

(b) To show that D is surjective we need to show that for every $g \in C^\infty$ there exists $f \in C^\infty$ such that $f' + af = g$. Multiplying by e^{ax} we get $f'e^{ax} + ae^{ax}f = ge^{ax}$. Then integrating both sides we get $e^{ax}f = \int ge^{ax}dx$. This suggests that we can define f as $\frac{\int ge^{ax}dx}{e^{ax}} = \left(\int ge^{ax}dx\right)e^{-ax}$ and then we obtain that $f' + af = g$. We only need to check that thus defined $f \in C^\infty$. Since $e^{-ax}, g, e^{ax} \in C^\infty$ we only need to check that the integral of ge^{ax} exists. But this is the case since continuous functions are differentiable and both e^{ax} and g are continuous, and thus ge^{ax} is continuous, so differentiable. \square