Problem Set 4, Part E – Solutions

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Solution   This is an "if and only if" problem so the proof must contain both implications.

"$\Rightarrow$"   We know that L is invertible and we want to prove that $ad - bc \neq 0$.

L is invertible means that L is a bijective function; i.e. L is both injective (1-to-1) and surjective (onto). In terms of the kernel and image of a linear transformation, this means that:

- L is injective $\Rightarrow Ker(L) = \{0\}$
- L is surjective $\Rightarrow Im(L) = F^2$

But, $Ker(L) = \{(x, y) | (ax + by, cx + dy) = (0, 0)\} = \{(x, y) | ax + by = 0 \text{ and } cx + dy = 0\}$

Now, solve this system of two equations with two unknowns: multiply the first equation by d and the second equation by b and subtract them $\Rightarrow x(ad - bc) = 0$. Since we are in the case when L is injective, which imples that $Ker(L) = 0$, we must have that the solution to the previous equation is just $x = 0$. For this to happen, we need to have $ad - bc \neq 0$.

Assume $ad - bc = 0$; then the equation is satisfied for all x, including for $x = 1$ and now if we return to our system, we can also solve it and obtain some value for y. So now this vector $(1, y) \in Ker(L) \Rightarrow$ the Kernel is not trivial. Contradiction.

So, we must have $ad - bc \neq 0$.

"$\Leftarrow$"   We know that $ad - bc \neq 0$ and we want to prove that L is bijective. So we need to prove that it's both injective and surjective.

Let's write once more what the $Ker(L)$ is: $Ker(L) = \{(x, y) | ax + by = 0 \text{ and } cx + dy = 0\}$

Analyzing again this system of equations as we did before, we get that $x(ad - bc) = 0$ and since $ad - bc \neq 0 \Rightarrow x = 0$. Returning to the system, we get $by = 0$ and $dy = 0$. But since $ad - bc \neq 0$, we can't have both $b = 0$ and $d = 0$ so one of them must be non-zero $\Rightarrow y$ must be 0.

So, the only solution of this system under these circumstances is $(0, 0) \Rightarrow Ker(L) = \{0\} \Rightarrow L$ is injective. We also obtain that $\dim(Ker(L)) = 0$.

Now we use the Rank-Nullity Theorem and obtain:

$$\dim(F^2) = \dim(Ker(L)) + \dim(Im(L)) \Rightarrow 2 = 0 + \dim(Im(L)) \Rightarrow \dim(Im(L)) = 2$$

But we know that $Im(L)$ is a subspace of the range of L, which is F^2. Since they have the same dimension by what we proved above $\Rightarrow Im(L) = F^2 \Rightarrow L$ is surjective.

So we proved that L is both injective and surjective $\Rightarrow L$ is bijective $\Rightarrow$ it is invertible.