Solution for HW9, part D
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Problem 1

Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of S is $\frac{16}{3}$.

Solution

First off, S is symmetric about the xz and yz and xy planes (we can reflect the shape about any of those planes without changing the set S) so the total volume is 8 times the volume in the first octant. For a given x, y, $0 \leq x$, $y \leq 1$, the height of the boundary is given by $min(\sqrt{1-x^2}, \sqrt{1-y^2})$. Now, $\sqrt{1-x^2} \leq \sqrt{1-y^2}$ iff $y \leq x$ (remember we're in the 1st octant (I have a strange feeling I'm just making this word up)). So when $y \leq x$, the roof of S is the cylinder which runs parallel to the x-axis and when $x \leq y$, the roof is the cylinder running parallel to the y-axis. Swapping x and y doesn't change the height of the roof, so we can restrict our attention to half of the first octant. That is, we can integrate the roof function over the triangle bounded by $x = 0$, $x = y$ and $y = 1$.

$$\frac{v(S)}{16} = \int_0^1 \int_0^x \sqrt{1-x^2}\,dydx.$$

This computes to 1/3, so the total volume is 16/3, as claimed.

Problem 2

Show that the volume of $B_1(0)$ in $\mathbb{R}^4$ is $\pi^2/2$.

Solution

We use the fact that the volume of a ball of radius r in $\mathbb{R}^3$ is $\frac{4}{3}\pi r^3$. But first an analogy with something that's easier to see: picture a ball in $\mathbb{R}^3$. If you squint really hard you can see that it's really a bunch of two dimensional balls stacked on top of one-another with radii depending on their height above ground. Analogously, in 4 dimensions, $B_1(0)$ is a bunch of 3-dimensional balls (infinitely many) stacked one on top of the other. The radius of a 3-ball at height z is $\sqrt{1-z^2}$. So the total volume is

$$2 \cdot \int_0^1 \frac{4}{3}\pi(1-z^2)^{3/2}dz =$$

$$\frac{8\pi}{3}\left[\frac{3}{8}\sin^{-1}(z) - \frac{z}{8}(2z^2-5)\sqrt{1-z^2}\right]_0^1$$

(I just looked this up to get the antiderivative). The second summand is 0 at both $z = 0$, $z = 1$ and the entire expression evaluates to $\pi^2/2$.