6. In class, we considered the set $A = \mathbf{Q} \cap [0,1]$ and showed that is has measure zero. In particular, we showed that it was countable, that is, we could write it as $A = \{a_0, a_1, a_2, a_3, \ldots\}$. Given any $\epsilon > 0$, we then covered it with rectangles $I_i = [a_i - \frac{\epsilon}{2^{i+2}}, a_i + \frac{\epsilon}{2^{i+2}}]$, for all $i \in \mathbf{N}$, so that

$$v(\bigcup_{i=0}^{\infty} I_i) \leq \sum_{i=0}^{\infty} v(I_i) = \epsilon$$

Fix $\epsilon = \frac{1}{2}$, and let $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$ be the open rectangle equal to the interior of the corresponding I_i defined above. Let $B = \cup_{i=2}^{\infty} J_i$ (*Note that we are purposefully omitting the first two sets, which cover 0 and 1 respectively, so that each* $J_i \subset [0,1]$*!*)

a. Show that $\partial B = [0,1] \setminus B$.

We recall from a previous problem set that $\partial B = \overline{B} \setminus B^\circ$. Clearly $B = B^\circ$ because B is a union of open sets and is thus open. To show that $\overline{B} = [0,1]$ we make the following assertion. First, we note that $B \supset (\mathbf{Q} \cap (0,1))$. We recall that the rationals are dense in the reals so it is clear that $\overline{B} \supset (0,1)$. For completeness we also check that 0 and 1 can be obtained as limits of sequences of elements in B. But clearly this is true: consider for example $x_n = \frac{1}{n}$ and $y_n = \frac{n-1}{n}$. Clearly $x_n \to 0$ and $y_n \to 1$. So $\overline{B} = [0,1]$ and $\partial B = [0,1] \setminus B$ as desired.

b. Show that ∂B does not have measure zero.

Because $\partial B = [0,1] \setminus B$ it is clear that $[0,1] = B \cup \partial B$ and these sets are disjoint. Now assume that ∂B has measure zero. This means for any $\epsilon > 0$ we can cover ∂B with closed intervals $[a_i, b_i]$ in such a way that $\sum_i |b_i - a_i| < \epsilon$.

We can cover B as well using the intervals $I_2, I_3, \ldots$. This gives us a cover of B such that the sum of the lengths of these intervals is $\sum_{j=2}^{\infty} \frac{1}{2^{j+2}} = \frac{1}{8}$.

But we note that the union of these covers is a cover for $[0,1]$ (because every point in $[0,1]$ is either in B or in ∂B). Thus we have found a cover for $[0,1]$ with measure less than $\frac{1}{8} + \epsilon$. For sufficiently small ϵ this is clearly less than 1, the content of $[0,1]$. But this is a contradiction, so ∂B cannot have measure 0.

c. Let χ_B be the characteristic function of B. Show that χ_B is not integrable on $[0,1]$.

(*Note that although B is a "reasonable" set in the sense that it is the union of a countable collection of open sets, it does not have a "reasonable" boundary, and so* χ_B *is not integrable.*)

We know from class that a function is not integrable if the set of its discontinuities has nonzero measure. But by definition the set of discontinuities for the characteristic function χ_B is precisely ∂B. So using part b we can conclude that χ_B is not integrable on $[0,1]$.