Problem Set 1, Part A – Solutions

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Problem 2 Since $\{c_n\}$ converges to c, this means that $\forall \epsilon > 0$ there exists a natural number N such that $|c_n - c| < \epsilon$ for all $n > N$. Let $\epsilon = 1 \Rightarrow$ we can find a natural numaber N with the above property. Therefore,

$$|c_n - c| < 1 \Rightarrow |c_n| - |c| < 1 \Rightarrow |c_n| < |c| + 1$$

for all $n > N$. So, $|c_n|$ is bounded for all $n > N$. But the only left terms are $\{c_1, ..., c_N\}$ and of cours, the absolute value of these terms is bounded by the one with the greatest absolute value $\Rightarrow$

$$|c_n| \leq max\{|c| + 1, |c_1|, ...|c_N|\}$$

□

Problem 6 We showed in class that if f and h are both continuous at a, then $f \cdot h$ is continuos at a. So, let $h = 1/g$ and so the only thing we need to prove is that $1/g$ is continuous at a, given that g is continuous at a and $g(a) \neq 0$.

For this, it is enough to show that $\forall \epsilon > 0$, there exists $\delta > 0$ such that $\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| < \epsilon$, whenever $|x - a| < \delta$.

$$\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| = \frac{|g(x) - g(a)|}{|g(x)| \cdot |g(a)|}$$

Since g is continuous, we get thet $\forall \epsilon' > 0$ there exists $\delta' > 0$ such that $|g(x) - g(a)| < \epsilon'$, whenever $|x - a| < \delta'$.

Also, since $g(a) \neq 0$, this means that there exists $M > 0$ such that $|g(a)| > M$. Now, in the above continuity property, let $\epsilon' = M/2 \Rightarrow$ there exists δ'' such that $|g(x) - g(a)| < M/2 \Rightarrow$ by triangle inequality, $|g(a)| - |g(x)| < M/2 \Rightarrow |g(x)| > M - M/2 = M/2$, for all x such that $|x - a| < \delta''$.

Now, in the definition of continuity of g, let $\epsilon' = \frac{2\epsilon M^2}{2}$. Then, retruning to the continuity of $1/g$ we get that:

$$\frac{|g(x) - g(a)|}{|g(a)| \cdot |g(x)|} < \frac{\epsilon M^2}{2} \cdot \frac{1}{M} \cdot \frac{2}{M} < \epsilon$$

for $\delta = min(\delta', \delta'')$

□