3. a. Let S be any set, and define $d(x,x) = 0 \forall x \in S$ and $d(x,y) = 1$ if $x \neq y$. Show that (S,d) is a metric space. For the record, this particular metric is known as the *discrete metric*.

To show that (S,d) is a metric space we need to check that d is positive, definite, symmetric, and satisfies the triangle inequality. Clearly $d(x,y) \geq 0$ for all $x, y \in S$. Additionally, the definition of d guarantees that $d(x,y) = 0$ iff $x = y$. Symmetry of d follows from symmetry of the relations $=$ and $\neq$. So the only non-trivial property is the triangle inequality.

Let $x, y, z \in S$. We wish to show that $d(x,y) \leq d(x,z) + d(z,y)$. We consider two cases: $x = y$ and $x \neq y$. If $x = y$ then $d(x,y) = 0$. Because d is positive, the triangle inequality must hold. Now assume $x \neq y$. In order for the triangle inequality to be true we must have either $x \neq z$ or $z \neq y$. But if both these statements are false (i.e. $x = z$ and $z = y$) then we must have $x = y$ because $=$ is an equivalence relation and thus is transitive. But this contradictions our assumption, so the triangle inequality holds in this case as well. So we see that (S,d) is a metric space.

b. Suppose S is a metric space with distance function d. Show that S is also a metric space with new distance function given by:

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Again, we must check that d' is positive, definite, symmetric, and satisfies the triangle inequality. But first, we check that d' is well defined. As d is a metric, it is positive so for all $x, y \in S$ the denominator $1 + d(x,y) > 0$. So to prove that d' is positive definite, it suffices to consider the numerator, and these two properties follow immediately from the fact that d is positive definite. Similarly $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$ for all $x, y \in S$ because d is symmetric.

Again, the most difficult property is the triangle inequality. To prove this we note that the function $f(n) = \frac{n}{1+n}$ is monotonically increasing for $n \geq 0$. Thus if $n > m$, $f(n) > f(m)$. So it follows from the fact that $d(x,y) \leq d(x,z) + d(z,y)$ that

$$\frac{d(x,y)}{1 + d(x,y)} \leq \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)}$$

for all $x, y, z \in S$. So

$$\frac{d(x,y)}{1 + d(x,y)} \leq \frac{d(x,z)}{1 + d(x,z) + d(z,y)} + \frac{d(z,y)}{1 + d(x,z) + d(z,y)} \leq \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)}$$

as d is positive. So d' satisfies the triangle inequality and (S, d') is a metric space.

4. Consider $S = \mathbf{R}^2$, with $v = (a,b)$ and $w = (c,e)$. We define the *Memphis metric* by $d(v,v) = 0$ for any v, and for $v \neq w$ by:

$$d(v,w) = \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2}$$

a. Show that (S, d) is a metric space.

As before, we must show that d is positive, definite, symmetric, and satisfies the triangle inequality. We begin by noting that for any $v \in \mathbf{R}^2$ the function $||v|| = \sqrt{a^2 + b^2}$ is positive. It is clear that the sum of two positive functions is positive definite, so d is as well. If $v \neq w$ than at least one of v and w is nonzero, so it is clear that $d(v, w) > 0$. By definition $d(v, v) = 0$ for all $v \in S$ so d is definite. Clearly $d(v, w) = \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2} = \sqrt{c^2 + e^2} + \sqrt{a^2 + b^2} = d(w, v)$ so d is symmetric.

Once again, the triangle inequality is the only non-trivial case. We let $u, v, w \in S$. Because d is defined as a piecewise, function, we need to consider two cases: $u = v$ and $u \neq v$. If $u = v$ then $d(u, v) = 0$ so it follows from positivity of d that $d(u, v) \geq d(u, w) + d(w, v)$. If $u \neq v$ then $d(u, v) = ||u|| + ||v||$ with the Euclidean norm. As in part 3 a, we cannot have $u = wandw = v$, though one of these may be true. Assume wlog that $u = w$. Then $d(u, w) + d(w, v) = ||w|| + ||v|| = d(u, v)$ and the triangle inequality is satisfied. If $u \neq w \neq v$, then $d(u, w) + d(w, v) = ||u|| + 2||w|| + ||v|| \geq d(u, v)$ because the norm is positive. Thus the triangle inequality holds and (S, d) is a metric.

b. Find $B_\epsilon(0)$.

By definition $B_\epsilon(0) = \{v \in S : d(v, 0) < \epsilon\} = \{v \in S : ||v|| < \epsilon\}$. This is equivalent to the ball of radius ϵ about zero in the Euclidean metric usually associated with $\mathbf{R}^2$.

c. For $v \neq 0$, find $B_\epsilon(v)$, for various $\epsilon > 0$.

For fixed v, by definition $B_\epsilon(v) = \{w \in S : d(v, w) < \epsilon\} = \{w \in S : ||w|| < \epsilon - ||v||\} \cup \{v\}$. We note that with the second definition it is necessary to include v because $d(v, v) = 0$ for any metric so v is always in $B_\epsilon(v)$. We describe more specifically what happens to $B_\epsilon(v)$ in two cases: $\epsilon \leq ||v||$ and $\epsilon > ||v||$. If $\epsilon \leq ||v||$ then $\epsilon - ||v|| \leq 0$. Because the norm function is positive, there are no $w \in S$ such that $||w|| < 0$. So $B_\epsilon(v) = \{v\}$ in this case.

If $\epsilon > ||v||$ then the difference $\epsilon - ||v||$ is positive. Let $\epsilon' = \epsilon - ||v||$. Then $B_\epsilon(v)$ corresponds to the ball of radius ϵ' about the origin with respect to the Euclidean metric, plus the point v.