Solution for HW1, part D
Geoff Anderson
granders@fas.harvard.edu

Problem 7

Given a set S in a normed vector space, we define a point $x \in S$ to be an *interior point* if $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset S$. We define the *interior* of S to be the set of interior points, and we denote it by S^o. Show that the interior of any set is open.

Solution We need to show that given $x \in S^o$, there exists $B_\epsilon(x) \subset S^o$. Let $x \in S^o$. Then by definition $\exists B_\epsilon(x) \subset S$. Now $B_\epsilon(x)$ is open, so $\forall y \in B_\epsilon(x)$, there exists $B_\delta(y) \subset B_\epsilon(x) \subset S$. Hence every element of $B_\epsilon(x)$ is in S^o, i.e. $B_\epsilon(x) \subset S^o$ and we're done.

Problem 11

Let $\{S_n\}$ be a collection of open sets in a normed vector space, and let $\{T_n\}$ be a collection of closed sets. Show that:

(a) $S_1 \cap S_2$ is open.

(b) $\bigcup S_n$ is open.

(c) $T_1 \cup T_2$ is closed.

(d) $\bigcap T_n$ is closed.

Solution

(a) Given $x \in S_1 \cap S_2$ there exist ϵ_1, ϵ_2 such that $B_{\epsilon_1}(x) \subset S_1$ and such that $B_{\epsilon_2} \subset S_2$. The smaller of the two epsilon-balls is contained in the $S_1 \cap S_2$. Hence $\forall x \in S_1 \cap S_2$ there is an epsilon-ball neighborhood of x which is a subset of $S_1 \cap S_2$, implying $S_1 \cap S_2$ is open.

(b) Just observe that $x \in \bigcup S_n$ implies $\exists S_i \in \{S_n\}$ such that $x \in S_i$. Since S_i is open, $\exists B_\epsilon(x) \subset S_i \subset \bigcup S_n$. Note that $\bigcup S_n$ is not notation for union from 1 to n. n here is the index variable, so we're taking the union of all the sets contained in the collection $\{S_n\}$. This collection can be infinite and needn't even be countable. The proof still holds.

(c), (d) Use the facts that $(A \cup B)^c = A^c \cap B^c$ and $(\bigcap A_n)^c = \bigcup A_n^c$ and apply (a) and (b).