6. A subset S in a metric space or a normed vector space is called *discrete* if, for every $x \in S$, there is some $\epsilon > 0$ such that $B_\epsilon(x) \cap S = \{x\}$, that is, the only intersection between the ball and the set is the point itself.

a. Show that every $f : S \to \mathbf{R}$ is continuous if S is discrete.

By definition, f is continuous if for every open subset $U \subset \mathbf{R}$, $f^{-1}(U)$ is open in S. Let $x \in f^{-1}(U)$. Because S is discrete, there exists some ϵ such that $B_\epsilon(x) \cap S = \{x\}$. But this means that $\{x\}$ is open. (By definition, a set B is open in a subset S of a metric space iff there exists some open $A \subset X$ such that $B = A \cap S$.) So $x \in \{x\} \subset f^{-1}(U)$ which means that $f^{-1}(U)$ is open. Thus f is continuous.

b. Consider $\mathbf{R}^n$ with its usual inner product. Show that every closed, bounded, and discrete set is finite, and give examples why each of these three conditions is necessary.

First let's start with the examples. The set $\{\frac{1}{n} : n \in \mathbf{Z}\} \subset \mathbf{R}$ is discrete and bounded but not closed. The set $\mathbf{Z} \subset \mathbf{R}$ is closed (it's complement is open) and discrete as we'll see below, but not bounded. And the set $[0, 1] \subset \mathbf{R}$ is closed and bounded but not discrete. Each of these sets are infinite so all three conditions must be necessary.

Now we begin the proof. Let $S \subset \mathbf{R}^n$ be closed, bounded, and discrete. Assume that $|S|$ is infinite. As S is bounded there exists $M > 0$ such that $S \subset [-M, M]^n$ a n-dimensional box with "radius" M. We consider the canonical integer lattice in $\mathbf{R}^n$: namely, the set of vectors in $\mathbf{R}^n$ that can be written as an integer linear combination of the standard basis vectors. As M is finite, there exist a large but finite number of such lattice points inside the box $B = [-M, M]^n$. About each lattice point $x \in \mathbf{R}^n$ we construct a box $B_x = [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x - n + 1]$. It is clear that these boxes cover the set B and thus S. As there exist a finite number of boxes that cover an infinite set, we know that one of the boxes must contain an infinite number of points of S. Call this box B_1.

We now repeat the above procedure by considering the $\frac{1}{2}$ integer lattice in B_1 (i.e., the set of points that are integer or half-integer linear combinations of the standard basis vectors). About each lattice point, we construct a box of radius $\frac{1}{2}$ in the same manner as above and in this way obtain a finite cover for B_1. Thus there exists some box, which we'll call B_2 which contains an infinite number of points of S. We proceed by considering the $\frac{1}{4}$ integer lattice in B_2 and so on.

Finally, we construct a Cauchy sequence $\{s_n\} \subset S$ by choosing some point $s_n \in B_n \cap S$ for each $n \in \mathbf{N}$. Clearly this sequence is Cauchy because for each $N \in \mathbf{N}$, $|s_n - s_m| \leq \frac{1}{2N}$ for all $n, m \geq N$ because $s_n, s_m \in B_N$. Because $S \subset \mathbf{R}^n$ which is a complete vector space, every Cauchy sequence converges and there exists some $s \in \mathbf{R}^n$ such that $s_n \to s$ as $n \to \infty$ (if you are not convinced, consider each coordinate of each vector separately). Because S is closed and $\{s_n\} \subset S$, the point $s \in S$.

Because S is discrete, there must exist some $\epsilon > 0$ such that $B_\epsilon(s) \cap S = \{s\}$. But this contradicts the fact that $s_n \to s$. This contradiction shows that $|S|$ is indeed finite.

c. Show that $\mathbf{Z} \subset \mathbf{R}$ is discrete.

Let $\epsilon = \frac{1}{2}$. Then for each integer $n \in \mathbf{Z}$, $B_{1/2}(n) \cap \mathbf{Z} = \{n\}$. So it is clear that $\mathbf{Z} \subset \mathbf{R}$ is discrete.