Solution for HW3, part B

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Problem 6

Show that $SL_n(\mathbb{R})$ is closed but not compact.

Solution

$SL_n = \{M \in M_n(\mathbb{R}) | det(M) = 1\}$. Now, recalling our identification of $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ we can think of $det : M_n(\mathbb{R}) \to \mathbb{R}$ as a polynomial (homogeneous of degree n) in n^2 variables. It is therefore continuous, since sums and products of continuous functions are continuous.

$\{1\} \subset \mathbb{R}$ is a closed subset of $\mathbb{R}$, so $SL_n(\mathbb{R}) = det^{-1}(\{1\})$ is closed.

SL_n is not compact because it is not bounded. There are lots of sequences of matrices in SL_n which are unbounded. For example,

$$A_m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m & 0 & \dots & 1 \end{pmatrix}$$

$\| A_m \|^2 = n + m^2$ for all m, so this sequence is unbounded as claimed.

Problem 7

Show that $O_n(\mathbb{R})$ is compact by showing that:

(a) $O_n(\mathbb{R})$ is closed.

(b) $O_n(\mathbb{R})$ is bounded.

Solution

Recall that $O_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) | M^t M = I\}$. Consider the map $f : M_n \to M_n$ which sends $A \in M_n$ to $f(A) = A^t A$. Each entry of the matrix $f(A)$ is a polynomial (homogeneous, of degree 2) in the matrix entries of A, so f is continuous.

The set $\{I\} \subset M_n$ is a single point in $M_n = \mathbb{R}^{n^2}$ and therefore closed. $O_n = f^{-1}(\{I\})$ is therefore closed, since f is continuous.

Let $A \in O_n$. If we denote the ith column vector by A_i, note that the ijth entry of $A^t A$ is $< A_i, A_j >$ the inner product of A_i and A_j. Therefore, since $A^t A = I$, the column vectors of A are an orthonormal collection of vectors in $\mathbb{R}^n$. Now $\| A \|^2 = \sum_{i,j} A_{ij}^2 = \sum_i \| A_i \|^2 = n$. This holds for any $A \in O_n$, so O_n is certainly bounded.