Problem Set 3, Part C – Solutions

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Problem 8 Since $M_n(\mathbb{R})$ is the whole space of matrices, then clearly, $\overline{GL_n(\mathbb{R})} \subset M_n(\mathbb{R})$, so the only thing we need to prove is that $M_n(\mathbb{R}) \subset \overline{GL_n(\mathbb{R})}$. For that, it is enough to show that given any matrix $A \in M_n(\mathbb{R})$, there exists a sequence $A_m \in GL_n(\mathbb{R})$ of matrices which converges to A.

- if $A \in GL_n(\mathbb{R})$, then just take the constant sequence $A_m = A$; it satisfies the properties;
- if $A \notin GL_n(\mathbb{R})$, then $\det(A) = 0$. Now take the sequence of matrices $A_m = A - \frac{1}{m} \cdot I_n$. Clearly, $\{A_m\} \to A$. Moreover, the determinant of A_m is equal to zero only for finitely many m since this is in fact the characteristic polynomial of A, evaluated at $1/m$ – this is because the characteristic polynomial has degree n, which means that it has at most n real roots, so $det(A - 1/m \cdot I_n) = 0$ only for at most n values of m. Then, we can simply throw away those matrices from the sequence and since there are finitely many of them, we will still be left with a sequence converging to A, but this time, all matrices from the sequence have non-zero determinat $\Rightarrow$ they are from $GL_n(\mathbb{R})$. This is the sequence we were looking for.

In conclusion, the closure of $GL_n(\mathbb{R})$ is $M_n(\mathbb{R})$.

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