8. Let X be the collection of diagonalizable matrices in $M_2(\mathbf{R})$. Is X open, closed, or neither? Explain.

We will show that X is neither open nor closed. From the Math 23a Final we have a good characterization of diagonalizable matrices: namely, for every matrix $A \in M_2(\mathbf{R})$ there either exists a basis so that A is of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or there exists a basis so that A is of the form $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$. The first case is clearly the diagonalizable matrices and the second case is the non-diagonalizable matrices. Note that in the first case a and/or b may be zero without changing the fact that A is diagonalizable. Note also that in the second case it is important that the upper left hand and lower right hand entries must be equal. The matrix $\begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$ has two distinct eigenvalues when $a \neq d$. Thus it has linearly independent eigenvalues and is diagonalizable.

Now we begin work with the problem at hand. Recall that a set is closed if it contains all of its limit points. Thus we show that X and X^C are not closed by constructing sequences in each set that converge to a point that it not in the set. For the first example we let $B_n = \begin{bmatrix} 1 & 1 \\ 0 & 1+\frac{1}{n} \end{bmatrix}$ for all $n \in \mathbf{N}$. It is clear that each $B_n \in X$ because its characteristic polynomial has two distinct roots and thus B_n has two linearly independent eigenvectors. We see that $B_n \to B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as $n \to \infty$ and we know from the Jordan blocks problem on the final that $B \notin X$. Thus X is not closed.

Next we consider a sequence of matrices $D_n = \begin{bmatrix} 1 & \frac{1}{n} \\ 0 & 1 \end{bmatrix}$ for all $n \in \mathbf{N}$. It is clear that each $D_n \in X^C$ because the matrix nD_n is in the Jordan blocks form and thus is not diagonalizable (so D_n can't be either). However, we see that $D_n \to D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as $n \to \infty$, and D is clearly a diagonal matrix. So X^C is not closed which means that X is not open. Thus X is neither open nor closed.