2.a. Let $f : \mathbf{R} \to \mathbf{R}^n$ and $g : \mathbf{R} \to \mathbf{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbf{R}$. Suppose that $\vec{p} = f(s_0)$ and $\vec{q} = g(t_0)$ are closer than any other pairs of points on the two curves. Prove that the vector $\vec{p} - \vec{q}$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$. (Hint: The point (s_0, t_0) must be a critical point for the function $\rho : \mathbf{R}^2 \to \mathbf{R}$ defined by $\rho : (s,t) = |f(s) - g(t)|^2$.)

We define $h(s,t) = |f(s) - g(t)|^2$. It is clear from our knowledge of calculus that the point (s_0, t_0) is a critical point of this function, as it is its minimal value. So the partial derivatives $(D_1 h)(s_0, t_0)$ and $(D_2 h)(s_0, t_0)$ must both be zero.

We compute using the chain rule $(D_1 h)(s_0, t_0) = 2(f(s_0) - g(t_0))f'(s_0) = 2(\vec{p} - \vec{q}) \cdot f'(s_0) = 0$. The expression $(\vec{p} - \vec{q}) \cdot f'(s_0) = 0$ represents the dot product so $\vec{p} - \vec{q}$ is orthogonal to the velocity vector $f'(s_0)$. Similarly $(D_2 h)(s_0, t_0) = -2(f(s_0) - g(t_0))g'(t_0) = -2(\vec{p} - \vec{q})g'(t_0) = 0$ so $\vec{p} - \vec{q}$ is orthogonal to the velocity vector $g'(t_0)$.

b. Apply the result of part (a) to find the closest pair of points of the "skew" straight lines in $\mathbf{R}^3$ defined by $f(s) = (s, 2s, -s)$ and $g(t) = (t+1, t-2, 2t+3)$.

We begin by computing the derivatives $f'(s) = (1, 2, -1)$ and $g'(t) = (1, 1, 2)$. Thus we wish to find the values of s and t so that $\vec{v} = (s - t - 1, 2s - t + 2, -s - 2t - 3)$ is orthogonal to these vectors. Thus we compute $\vec{v} \cdot f'(s) = s - t - 1 + 4s - 2t + 4 + s + 2t + 3 = 6s - t + 6$ and $\vec{v} \cdot g'(t) = s - t - 1 + 2s - t + 2 - 2s - 4t - 6 = s - 6t - 5$. So $6s - t + 6 = 0 = s - 6t - 5$ which means that $s = \frac{-41}{35}$ and $t = \frac{-36}{35}$. So the closed pair of points on these lines are $f(\frac{-41}{35}) = (\frac{-41}{35}, \frac{-82}{35}, \frac{41}{35})$ and $g(\frac{-36}{35}) = (\frac{-1}{35}, \frac{-106}{35}, \frac{33}{35})$.

3. Consider a particle which moves on a circular helix in $\mathbf{R}^3$ with position vector given by (all scalars non-zero):

$$\phi(t) = (a\cos\omega t, a\sin\omega t, b\omega t).$$

b. Show that the speed of the particle is constant.

The speed of the particle is defined to be the norm of the derivative. Thus we compute $\phi'(t) = (-a\omega\sin\omega t, a\omega\cos\omega t, b\omega)$. So $|\phi'(t)| = \sqrt{(-a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2 + (b\omega)^2} = \sqrt{a^2\omega^2 + b^2\omega^2}$. Clearly this is a constant for all values of t.

c. Show that the velocity vector makes a constant nonzero angle with the z-axis.

From part b, we know that $\phi'(t) = (-a\omega\sin\omega t, a\omega\cos\omega t, b\omega)$. We compute the dot product $\phi'(t) \cdot (0,0,1) = b\omega$. We know from Math 23a that the angle between two vectors $\vec{v}$ and $\vec{w}$ is defined by $\cos\theta = \frac{<\vec{v}, \vec{w}>}{||\vec{v}|| \cdot ||\vec{w}||}$. So

$$\cos\theta = \frac{b\omega}{\sqrt{a^2\omega^2 + b^2\omega^2}}$$

which is constant. Furthermore, we know that $\theta \neq 0$ if $\cos\theta \neq 1$. But $a, b, \omega \neq 0$ so it is clear that the expression on the right hand side cannot equal one. Thus θ is a constant non-zero angle as required.

d. If $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$, notice that $\phi(t_1) = (a, 0, 0)$ and $\phi(t_2) = (a, 0, 2\pi b)$, so the vector $\phi(t_2) - \phi(t_1)$ is vertical. Conclude that the equation

$$\phi(t_2) - \phi(t_1) = (t_2 - t_1)\phi'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

As suggested, we notice that if $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$ then $\phi(t_2) - \phi(t_1) = (0, 0, 2\pi b)$ is parallel to the z-axis. Thus if there is some vector $\phi'(\tau)$ satisfying the equation $\phi(t_2) - \phi(t_1) = (t_2 - t_1)\phi'(\tau)$, this vector must be parallel to the z-axis. But we showed in part c that the velocity vectors of ϕ form a non-zero angle with the z-axis. Thus, no such vector is parallel to this axis and the Mean Value Theorem does not hold for vector-valued functions.