Solution for HW4, part C
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Problem 5

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0, y) = y^2$.

(a) Show that f is continuous at $(0, 0)$.

(b) Find the partial derivatives of f at $(0, 0)$.

(c) Show that f is differentiable at $(0, 0)$.

(d) Show that $D_1 f$ is *not* continuous at $(0, 0)$.

Solution

(a) Note $|f(x, y)| \leq |x^2 \sin(1/x)| + |y^2| \leq x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$ (this holds whether or not $x = 0$). Now use limit-point or epsilon-delta or whatever definition of continuity you'd like.

(b) $D_1 f = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} h \sin(1/h)$. Now, for all $h > 0$,

$$-h \leq h \sin(1/h) \leq h$$

Hence $D_1 f = 0$.

$D_2 f = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} h = 0$.

(c) The derivative exists at $(0, 0)$ and is in fact the 0 map. We check that this works: let $h = (h_1, h_2)$ and let L be the 0 map.

Then $\lim_{|h| \to 0} \frac{|f((0,0)+h) - f(0,0) - L(h)|}{|h|} = \lim_{|h| \to 0} |f(h_1, h_2)|/|h| \leq \lim_{|h| \to 0} (h_1^2 + h_2^2)/|h| = \lim_{|h| \to 0} |h| = 0$. So the 0 map is the derivative at $(0, 0)$, as claimed.

(d) When $x \neq 0$, $D_1 f(x, y) = 2x \sin(1/x) - \cos(1/x)$. In any neighborhood of $(0, 0)$, there are infinitely many points (x, y) such that $|D_1 f(x, y)| = 1$, e.g. all those of the form $(1/(\pi k), 0)$ for $k \in \mathbb{N}$ large enough. But $D_1 f(0, 0) = 0$, so $D_1 f$ is not continuous at $(0, 0)$.