Problem Set 5, Part A – Solutions

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Problem 3

(a) Define the function $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(r, \theta) = (r\cos\theta, r\sin\theta) = (x, y)$. Then, we have that $g(r, \theta) = f \circ h(r, \theta)$ and using the chain rule we get:

$$Jg = Jf \cdot Jh \Rightarrow$$

$$\begin{pmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & r\sin\theta \\ \\ sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta & -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta \end{pmatrix}$$

Having these relations, one only needs to plug in these values in the relation from the problem and do some simple computations.

□

(b) The second part can be similarly obtained:

$$\begin{aligned}
\frac{\partial^2 g}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\sin\theta = \\
&= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)\cos\theta = \\
&= \left(\frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial y \partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 f}{\partial x \partial y}\cos\theta + \frac{partial^2 f}{\partial y^2}\sin\theta\right)\sin\theta = \\
&= \frac{\partial^f}{\partial x^2}\cos^2\theta + \frac{\partial^f}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta
\end{aligned}$$

Similarly, we obtain:

$$\frac{\partial^2 g}{\partial \theta^2} = r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x \partial y} - r^2\sin\theta\cos\theta\frac{\partial^2 f}{\partial y \partial x} - r\sin\theta\frac{\partial f}{\partial y} - r\cos\theta\frac{\partial f}{\partial x}$$

Having found these relations, we just need to plug them in the relation from the statement and do the necessary cancellations.

□