4. Following up on problem #3 if $f : \mathbf{R}^n \to \mathbf{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

With f as above, we say that f is *harmonic* on the open set $U \subset \mathbf{R}^n$ provided that $\nabla^2 f(x) = 0$ for all $x \in U$.

a. Fine a (simple) condition on the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.

We easily compute $\frac{\partial^2 f}{\partial x^2} = 2a$ and $\frac{\partial^2 f}{\partial y^2} = 2c$. So $\nabla^2 f(x) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2a + 2c = 0$ iff $a + c = 0$.

b. Show that $f : \mathbf{R}^n \to \mathbf{R}$ defined by $f(x) = \frac{1}{||x||^{n-2}}$ is harmonic on $U = \mathbf{R}^n - \{0\}$.

We begin by computing the partial derivatives $\frac{\partial f}{\partial x_i} = (1 - \frac{n}{2})(x_1^2 + \cdots x_n^2)^{\frac{-n}{2}} 2x_i = (2 - n)x_i||x||^{-n}$ for $x \neq 0$. So $\frac{\partial^2 f}{\partial x_i^2} = (2 - n)||x||^{-n} + (2 - n)x_i \frac{-n}{2}(x_1^2 + \cdots + x_n^2)^{(\frac{-n}{2} - 1)} 2x_i = (2 - n)\frac{||x||^2 - nx_i^2}{||x||^{n+2}}$. Clearly this holds for any $i = 1, \ldots, n$ and any $x \in U$. We easily compute

$$\nabla^2 f(x) = \sum_{i=1}^{n} (2 - n)\frac{||x||^2 - nx_i^2}{||x||^{n+2}} = (2 - n)\frac{n||x||^2 - n||x||^2}{||x||^{n+2}} = 0$$

Thus f is harmonic for all $x \in \mathbf{R}^n - \{0\}$.

c. Show that if $g : \mathbf{R}^2 \to \mathbf{R}$ is harmonic, then $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.

This problem is done most easily if we interpret f as a composition of $g(z, w)$ with the function $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (e^x \cos y, e^x \sin y)$. Clearly $f = g \circ T$. So by the chain rule we have the following matrix equality:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{bmatrix} \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

We could also compute the equations $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial z} e^x \cos y + \frac{\partial g}{\partial w} e^x \sin y$ and $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial z} e^x \sin y + \frac{\partial g}{\partial w} e^x \cos y$ directly. We compute the second order derivatives by hand:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial g}{\partial z} e^x \cos y + \frac{\partial g}{\partial w} e^x \sin y\right]$$

$$= \frac{\partial^2 g}{\partial z^2}\frac{\partial z}{\partial x} e^x \cos y + \frac{\partial g}{\partial z} e^x \cos y + \frac{\partial^2 g}{\partial w^2}\frac{\partial w}{\partial x} e^x \sin y + \frac{\partial g}{\partial w} e^x \sin y$$

$$= \frac{\partial^2 g}{\partial z^2} e^{2x} \cos^2 y + \frac{\partial g}{\partial z} e^x \cos y + \frac{\partial^2 g}{\partial w^2} e^2 x \sin^2 y + \frac{\partial g}{\partial w} e^x \sin y$$

Similarly:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[-\frac{\partial g}{\partial z}e^x \sin y + \frac{\partial g}{\partial w}e^x \cos y\right]$$

$$= -\frac{\partial^2 g}{\partial z^2}\frac{\partial z}{\partial y}e^x \sin y + -\frac{\partial g}{\partial z}e^x \cos y + \frac{\partial^2 g}{\partial w^2}\frac{\partial w}{\partial y}e^x \cos y - \frac{\partial g}{\partial w}e^x \sin y$$

$$= \frac{\partial^2 g}{\partial z^2}e^{2x} \sin^2 y + -\frac{\partial g}{\partial z}e^x \cos y + \frac{\partial^2 g}{\partial w^2}e^{2x} \cos^2 y - \frac{\partial g}{\partial w}e^x \sin y$$

Therefore

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{2x}(\frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial w^2})(\cos^2 y + \sin^2 y) = e^{2x}(\frac{\partial^2 g}{\partial z^2} + \frac{\partial^2 g}{\partial w^2}) = 0$$

because g is harmonic. So f is harmonic as well.