Solution for HW6, part A
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Problem 3

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^3 + e^y$, and let $C = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$ be a level set of f.

(a) Show that for every point of C, there is a neighborhood of the point in which y may be defined implicitly as a function of x.

(b) Find $\frac{dy}{dx}$ at every point of C.

Solution

Computing the Jacobian of f, we find that $Jf = [\ 2x \quad 3y^2 + e^y\ ]$. $D_2 f = 3y^2 + e^y$, so it is nonzero for all $y \in \mathbb{R}$. Hence given any point $(x_0, y_0) \in \mathbb{R}^2$, we can apply the implict function theorem: there exist neighborhoods $A \ni x_0$ and $B \ni y_0$ such that for each $x \in A$ there is a unique $h(x) \in B$ with $f(x, h(x)) = 0$.

(b) $\frac{dy}{dx}$ can be computed with the formula derived in class, $\frac{dy}{dx} = -D_1 f / D_2 f$. That is,

$$\frac{dy}{dx} = \frac{-2x}{3y^2 + e^y}.$$

Problem 5

Consider the set S of points in $\mathbb{R}^5$ defined by the to equations:

$$f_1(x, y, z, u, v) = xu^2 + yzv + x^2 z - 3 = 0$$

$$f_2(x, y, z, u, v) = xyv^3 + 2zu - u^2 v^2 - 2 = 0$$

a) Show that there is a neighborhood of the point $(1, 1, 1, 1, 1) \in S$ and a differentiable function $h : \mathbb{R}^3 \to \mathbb{R}^2$ such that in the neighborhood, the point $(x, y, z, h(x, y, z)) \in S$ and

b) find $Jh(1, 1, 1)$.

Solution

Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be defined by $f(x, y, z, u, v) = (f_1(x, y, z, u, v), f_2(x, y, z, u, v))$.

$$\begin{bmatrix} D_u f_1 & D_v f_1 \\ D_u f_2 & D_v f_2 \end{bmatrix} = \begin{bmatrix} 2xu & yz \\ 2z - 2uv^2 & 3xyv^2 - 2u^2 v \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

(evaluated at $(1, 1, 1, 1, 1)$), which has nonzero determinant. Hence by the implicit function theorem, we may write (u, v) as a function of (x, y, z) in some neighborhood of $(1, 1, 1)$. I.e. $(u, v) = h(x, y, z)$.

b) The proof of the implicit function theorem suggests a general method to do this sort of calculation. As in said proof, let $F : \mathbb{R}^5 \to \mathbb{R}^5$ where $F(x, y, z, u, v) = (x, y, z, f(x, y, z, u, v))$.

F is locally invertible wherever $det JF$ is nonzero (and hence wherever Jf is nonzero), so it is locally invertible at $(1,1,1,1,1)$. Also, F^{-1} has the form $F^{-1}(x,y,z,a,b) = (x,y,z,H(x,y,z,a,b))$ where H is a map into $\mathbb{R}^2$ with domain equal to the domain of F (some neighborhood of $(1,1,1,1,1)$).

Again as in the proof of the implicit function theorem, the function $h(x,y,z)$ in part a) is in fact defined by $h(x,y,z) = H(x,y,z,0,0)$. This is easy to check: given any (x,y,z,u,v) in the locus of f, we have

$$(x,y,z,u,v) = F^{-1} \circ F(x,y,z,u,v) = F^{-1}(x,y,z,0,0) = (x,y,z,h(x,y,z)).$$

The point of all of this is that computing the Jacobian of h isn't too bad:

$$JF^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ D_x h_1 & D_y h_1 & D_z h_1 & * & * \\ D_x h_2 & D_y h_2 & D_z h_2 & * & * \end{bmatrix}.$$

So we just need to compute $JF(1,1,1,1,1)$ and then its inverse matrix and then look at the lower left hand entries to get $Jh(1,1,1)$.

Doing so, we obtain

$$JF(1,1,1,1,1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \end{bmatrix}$$

and then $Jh = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix}$.