Problem Set 6, Part B – Solutions

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Problem 6 (a) if $f : \mathbb{R} \to \mathbb{R}$ is locally invertible at every point (i.e. $f'(a) = 0, \forall a \in \mathbb{R}$), then assume that f is not one-to-one $\Rightarrow$ there must exist $x, y \in \mathbb{R}$ with $x \neq y$ such that $f(x) = f(y)$. Using the mean value theorem, we then get that there must exist $c \in (x, y)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y} = 0$ which contradicts the hypothesis.

□

(b) $g : \mathbb{R} \to \mathbb{R}, g(x, y) = (e^x \cos y, e^x \sin y)$. Then the Jacobian of g will look like:

$$\mathbf{J}[\mathbf{g}(\mathbf{x})] = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

Clearly, $det(J[g(x)]) = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x}$ which is always non-zero. Therefore, g is locally invertible at every point of its domain.

However, for $y = 0$ and $y = 2\pi$, we get that:

$g(x, 0) = (e^x, 0)$ and $g(x, 2\pi) = (e^x, 0)$. So, $g(x, 0) = g(x, 2\pi) \Rightarrow g$ is not one-to-one.

□