7. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Show that f is locally invertible in a neighborhood of every point except the origin, and compute f^{-1} explicitly.

By the Inverse Function theorem, to show that f is locally invertible at a point (x, y), it suffices to compute the determinant of the Jacobian and verify that it is nonzero. Thus when $(x, y) \neq (0, 0)$, we compute:

$$J(f, (x, y)) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{bmatrix}$$

So $|J(f, (x, y))| = \frac{y^2-x^2}{(x^2+y^2)^2}\frac{x^2-y^2}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2}\frac{-2xy}{(x^2+y^2)^2} = \frac{-((x^2-y^2)^2+4x^2y^2)}{(x^2+y^2)^4} = \frac{-(x^4+2x^2y^2+y^4)}{(x^2+y^2)^4} = \frac{-(x^2+y^2)^2}{(x^2+y^2)^4} = \frac{-1}{(x^2+y^2)^2}$. Clearly this expression is nonzero at all points other than the origin. So we are done.

8. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by: $f(x) = \frac{x}{2} + x^2 \sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$.

a. Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

We use the squeeze theorem to note that f is continuous at 0, which is a good sign. To show that f is differentiable at 0, we need to show that the following limit exists:

$$\lim_{h\to 0} \frac{\frac{h}{2} + h^2 \sin(\frac{1}{h})}{h}$$

(note this expression is considerably simplified since $f(0) = 0$). Clearly this equals

$$\lim_{h\to 0} \frac{1}{2} + h \sin(\frac{1}{h})$$

We note that the expression $\sin(\frac{1}{h})$ is bounded above by 1 and below by -1 regardless of what value h takes. So $-h \leq h \sin(\frac{1}{h}) \leq h$. We note that $\lim_{h\to 0} h = \lim_{h\to 0} -h = 0$, so by the squeeze theorem again,

$$\lim_{h\to 0} \frac{1}{2} + h \sin(\frac{1}{h}) = \frac{1}{2}$$

In particular, f is differentiable at 0 and $f'(0) = \frac{1}{2}$.

b. Show that there is no open set (interval) containing 0 on which f is one-to-one.

By a theorem proven in class, because f is differentiable at 0, we know that f is continuous in a neighborhood of the origin. You can check this directly, but it's a nice fact to know. So to show that there is no open interval about the origin on which f is one-to-one, it suffices to prove this for an arbitrary ϵ ball about 0, because each such interval contains an ϵ ball about 0 for some ϵ.

Let $\epsilon > 0$. We easily compute $f'(x) = \frac{1}{2} + 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$ for all $x \neq 0$. We see that for all $n \in \mathbf{N}$, $f'(\frac{1}{2\pi n}) = \frac{-1}{2}$ and $f'(\frac{1}{2\pi n+1}) = \frac{3}{2}$. As this is true for each $n \in \mathbf{N}$, we may choose

n large enough so that $\frac{1}{2\pi n} < \epsilon$. Thus, $\frac{1}{2\pi n}$ and $\frac{1}{2\pi n+1}$ give us two points within the interval $(-\epsilon, \epsilon)$ at which the derivative has different signs. Because f is continuous on $(-\epsilon, \epsilon)$ and is in fact continuously differentiable away from the origin, this means that f is not one-to-one on this interval. As ϵ was arbitrary, we have completed our proof.