
Solution for HW7, part B

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Problem 5

Let $q(x, y, z) = 2x^2 + 5y^2 + 2z^2 + 2xz$ be a quadratic form. Show that q is positive-definite by:

- (a) using Theorem 8.8 from Edwards
- (b) diagonalizing the quadratic form.

Solution

- (a) Really, just use Theorem 8.8.
- (b) The matrix for q in its symmetric form is

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Computing eigenvalues of A , we get 1, 3, 5. The spectral theorem tells us that since A is a symmetric matrix, there exists an orthonormal eigenbasis diagonalizing the quadratic form. So A can be diagonalized to

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Alternatively one could find such a basis using the usual methods. The first set of eigenvectors you find (one for each eigenvalue) will in fact be pairwise orthogonal. Why is this true? In general, given a symmetric $A \in M_n(\mathbb{R})$, the spectral theorem guarantees that the eigenspaces of A will span \mathbb{R}^n and that they will be pairwise orthogonal. That is, if $V_{a_1} \dots V_{a_k}$ are the eigenspaces of A , $A = \bigoplus_1^k V_{a_i}$ and given any v, w in distinct eigenspaces, $v \perp w$ (check it!).

Once we've found the eigenbasis, we just need to normalize the vectors. Doing so, we get $v_1 = (1/\sqrt{2})(1, 0, -1)$, $v_2 = (0, 1, 0)$, $v_3 = (1/\sqrt{2})(1, 0, 1)$.

Setting

$$B = \begin{bmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{bmatrix},$$

$A' = B^t \cdot A \cdot B$. Now

$$\left(B^t \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^t \cdot A' \cdot B^t \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = q(x, y, z)$$

(remember $B^t = B^{-1}$) so we get $q(x, y, z) = (1/\sqrt{2}(x-z))^2 + 5y^2 + 3(1/\sqrt{2}(x+z))^2$. Taadaa!

Some points of confusion: a number of people wrote that the matrix for q is

$$\begin{bmatrix} D_{xx}q & D_{xy}q & D_{xz}q \\ D_{yx}q & D_{yy}q & D_{yz}q \\ D_{zx}q & D_{zy}q & D_{zz}q \end{bmatrix}.$$

This is in fact the Hessian of q and is equal to $2 \cdot A$. However, the Hessian is positive definite iff A is, so for the purposes of this problem (or classifying critical points of a function) the Hessian is fine. Just make sure you know what you are doing one way or the other.

Another mistake: you can't diagonalize the matrix by doing row operations! Row operations don't result an equivalent matrix!