Problem Set 7, Part C – Solutions

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Problem 6 $f(x,y,z) = xy^2z^3, a = (1,0,-1)$.

$T_2(a+h)$ is given by the following formula:

$$T_2(a+h) = f(a) + (D_1f(a)\cdot h_1 + \ldots + D_3f(a)\cdot h_3) + \frac{1}{2}(D_{11}f(a)\cdot h_1^2 + 2D_{12}f(a)\cdot h_1h_2 + \ldots + D_{33}h_3^2)$$

and

$D_1 = y^2z^3, D_2 = 2xyz^3, D_3 = 3xy^2z^2, D_{11} = 0, D_{12} = 2yz^3 = D_{21}, D_{13} = 3y^2z^2, D_{22} = 2xz^3, D_{23} = 6xyz^2 = D_{32}, D_{33} = 6xy^2z$.

After calculations, $T_2(a+h) = -h_2^2 \Rightarrow R_2(h) = f(a+h) - T_2(h) = (1+h_1)h_2^2(-1+h_3)^3 + h_2^2$.

Then (solution thanks to Nancy Chen, Serena Rezny, Ana Batrachenko, Hilary Havens),

$$0 \le \left|\frac{R_2(h)}{\|h\|}\right| = \left|\frac{h_2^2((1+h_1)(h_3-1)^3+1)}{h_1^2+h_2^2+h_3^2}\right| = \left|\frac{h_2}{h_1^2+h_2^2+h_3^2}\right| \cdot \left|(1+h_1)(-1+h_3)^3+1\right|$$

$$\le \left|(1+h_1)(-1+h_3)^3+1\right| \to 0$$

as $\|h\| \to 0$. I used the fact that $\frac{h_2^2}{h_1^2+h_2^2+h_3^2} \le 1$.

Therefore, by the squeeze/sandwich principle,

$$\lim_{\|h\|\to 0} \frac{R_2(h)}{\|h\|} = 0$$

□