8. Let $f : \mathbf{R}^n \to \mathbf{R}$ be of class C^3. Let a be a critical point of f, that is, $\nabla f(a) = 0$. Show that if the quadratic form $q(h)$ corresponding to f at a is positive-definite, then $f(a)$ is a local minimum.

(We are looking for a δ - ϵ proof using the equation: $f(a+h) - f(a) = q(h) + R_2(h)$. You may use Taylor's Theorem without proof.)

We define $R_2(h) = f(a+h) - T_2(h)$ where $T_2(h) = f(a) + h\nabla f(a) + q(h)$ (note that here $q(h)$ includes the $\frac{1}{2}$ coefficient). By Taylor's Theorem we know that

$$\lim_{||h|| \to 0} \frac{R_2(h)}{||h||} = 0$$

Because $\nabla f(a) = 0$, we see that $f(a+h) - f(a) = q(h) + R_2(h)$. To show that $f(a)$ is a local minimum, we wish to show that there exists $\delta > 0$ such that $f(a+h) - f(a) > 0$ for all $||h|| < \delta$. So it suffices to show that $q(h) + R_2(h)$ is positive for small enough $h \neq 0$.

Clearly the expression $\frac{q(h)+R_2(h)}{||h||^2} > 0$ iff $q(h) + R_2(h) > 0$. And because q is a quadratic form $\frac{q(h)+R_2(h)}{||h||^2} = q(\frac{h}{||h||^2}) + \frac{R_2(h)}{||h^2||}$. We interpret $q(\frac{h}{||h||^2}) : \mathbf{R}^n - \{0\} \to \mathbf{R}$ as a function $\tilde{q}(x) : S^{n-1} \to \mathbf{R}$ because each $\frac{h}{||h||^2} \in S^{n-1}$. Furthermore because S^{n-1} is closed and bounded, it is compact, and because the function q and by extension $\tilde{q}$ is continuous, its image in $\mathbf{R}$ attains its minimum value. So $\tilde{q}(x) \geq m$ for all $x \in S^{n-1}$, and because q is positive definite $m > 0$. Thus $\frac{q(h)+R_2(h)}{||h||^2} \geq m + \frac{R_2(h)}{||h^2||}$. Now because $\lim_{||h|| \to 0} \frac{R_2(h)}{||h||} = 0$ we know there exists some $\delta > 0$ such that $| \frac{R_2(h)}{||h||} |< \frac{m}{2}$ for all $||h|| < \delta$. Thus for $h \in B_\delta(0)$, $\frac{q(h)+R_2(h)}{||h||^2} \geq m - \frac{m}{2} = \frac{m}{2} > 0$. Thus $f(a+h) - f(a) > 0$ in this δ neighborhood, and $f(a)$ is a local minimum as desired.