Solution for HW8, part C
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Problem 4

Let $f : A \to \mathbb{R}$ be integrable. Show that
(a)$|f|$ is integrable and that
(b) $\left|\int_A f\right| \leq \int_A |f|$.

Solution

I'll take an approach that relies on part of problem 5 and take the opportunity to sketch a proof. First, a few running assumptions: f is bounded and $A \subset \mathbb{R}^n$ is a closed rectangle.

Lemma

Let $I = \{f : A \to \mathbb{R}^n | f \text{ is integrable}\}$. Then I is a vector space over $\mathbb{R}$.

Pf(sketchy) of Lemma

I'll just show that I is closed under addition. Let $f, g \in I$. $f+g$ is integrable iff for all $\epsilon > 0$, there exists a partition P of A st $\sum_{P_i \in P}(M_{P_i}(f+g) - m_{P_i}(f+g)) \cdot v(P_i) < \epsilon$. Let B be any subset of A. Observe that for any $x \in B$, $(f+g)(x) \leq f(x) + \sup_B(g) \leq \sup_B(f) + \sup_B(g)$. Hence $\sup_B(f+g) \leq \sup_B(f) + \sup_B(g)$. Similarly, $\inf_B(f+g) \geq \inf_B(f) + \inf_B(g)$. So

$$M_B(f+g) - m_B(f+g) \leq (M_B(f) - m_B(f)) + (M_B(g) - m_B(g)).$$

Now using the integrability of f, g, integrability of $f+g$ follows right away. The rest of the proof I leave in *your* able hands.

Turning back to the original proof, we proceed by breaking up f into its positive and negative parts: let $f^+, f^- : A \to \mathbb{R}$ where

$$f^+(x) = \begin{cases} f(x) & \Leftrightarrow f(x) > 0 \\ 0 & \Leftrightarrow f(x) \leq 0 \end{cases}$$

and

$$f^-(x) = \begin{cases} f(x) & \Leftrightarrow f(x) < 0 \\ 0 & \Leftrightarrow f(x) \geq 0 \end{cases}$$

Now $f = f^+ + f^-$ and $|f| = f^+ - f^-$. If we show that f^+ and f^- are integrable, we can use our lemma and we'll be done! I'll show f^+ is integrable (f^- is analogous). Let $B \subset A$ and consider the three possible cases:

1. $M_B(f) > 0$, $m_B(f) > 0 \Rightarrow M_B(f^+) - m_B(f^+) = M_B(f) - m_B(f)$;
2. $M_B(f) > 0$, $m_B(f) \leq 0 \Rightarrow M_B(f^+) - m_B(f^+) \leq M_B(f) - m_B(f)$;
3. $M_B(f), m_B(f) \leq 0 \Rightarrow M_B(f^+) - m_B(f^+) = 0 \leq M_B(f) - m_B(f)$.

Hence for a partition P of A,

$$\sum_{P_i \in P} (M_{P_i}(f^+) - m_{P_i}(f^+)) \cdot v(P_i) \leq \sum_{P_i \in P} (M_{P_i}(f) - m_{P_i}(f)) \cdot v(P_i).$$

Integrability of f therefore implies integrability of f^+. f^- is similarly integrable and by our lemma, so is $|f|$.

As for (b), observe that for any partition P, $L(f, P) \leq L(|f|, P)$ and also $L(-f, P) = -L(f, P) \leq L(|f|, P)$. Taking the supremum of both sides of these equations, we get

$$\pm \int_A f \leq \int_A |f|,$$

or, saying the same thing,

$$\left| \int_A f \right| \leq \int_A |f|.$$

In case of notational confusion: $M_B(f) = \sup_B(f)$, $m_B(f) = \inf_B(f)$ (think "Maximum" and "minimum"). Also,

$$L(f, P) = \sum_{P_i \in P} m_{P_i}(f) \cdot v(P_i)$$

("Lower sum") and

$$U(f, P) = \sum_{P_i \in P} M_{P_i}(f) \cdot v(P_i)$$

("Upper sum").