6. For a function $f : [0,1] \to \mathbf{R}$, let $A = \{x \in [0,1] | f \text{ is not differentiable at } x\}$. Find such an f satisfying the following conditions: a. f is continuous, b. $f(0) = 0$, c. $f(1) = 1$, d. A has content zero, and e. if $x \notin A$, then $f'(x) = 0$.

We recall from class that the Cantor set consists of precisely those numbers in $[0,1]$ that can be written in a ternary expansion that contains no 1s. This gives us a binary expansion of sorts and we define a function f on the Cantor set A to make this explicit:

$$f(0.a_1a_2a_3\dots) = 0.b_1b_2b_3\dots \text{ where } b_k = \frac{1}{2}a_k$$

When the latter expression is interpreted as a binary decimal f maps A surjectively onto the unit interval. This map is not injective however and we note that $f(0.a_1\dots a_k0\overline{2}) = f(0.a_1\dots a_k2\overline{0}) = 0.b_1\dots b_k1\overline{0}$. We note that the values such that this is true correspond precisely to the endpoints of one of the $k+1$-th deleted intervals in the Cantor set. So we are inspired to extend f to the entire unit interval $[0,1]$ by defining f to be the common value shared by its endpoints on the interval $(0.a_1\dots a_k1, 0.a_1\dots a_k2)$.

We claim that this function has the desired properties. Clearly $f(0) = 0$ and $f(1) = 1$. Also it is clear that $f'(x) = 0$ for all $x \notin A$ because given some $x \notin A$ there is an open interval about x on which $f(x)$ is constant. Additionally, we know from class that A has content zero. So it remains to check that f is continuous on $[0,1]$ and that $f'(x)$ does not exist for all $x \in A$.

Clearly f is continuous on A^c (for one thing, it is differentiable on the complement of A). So it suffices to show that this is also true for $x \in A$. Let $\epsilon > 0$ and let $k \in \mathbf{N}$ be such that $\frac{1}{2^k} < \epsilon$. Then we note that for all $y \in B_{\frac{1}{3^k}}(x)$, the ternary expansion of y and x agree for the first $k-1$ digits. So from the definition of f, $|f(y) - f(x)| < \frac{1}{2^k} < \epsilon$ and f is a continuous function.

Finally say $x \in A$ and let $x_k \in A$ be such that x_k has the same decimal expansion as x except in the k-th decimal place (where one has the digit "0" and the other has the digit "2"). Then $|x - x_n| = \frac{1}{3^k}$ while $|f(x) - f(x_n)| = \frac{1}{2^k}$. So

$$\lim_{x_k \to x} \frac{f(x_k) - f(x)}{x_k - x} = \lim_{k\to\infty} (\frac{3}{2})^k = \infty$$

and f is not differentiable at x. Thus f satisfies the required conditions.