**S**olution for HW9, part D
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**Problem 8**

Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be two bounded functions such that the closure of $B = \{x \in A | f(x) \neq g(x)\}$ is a set of measure zero. (This implies that $f$ and $g$ agree except on a set of measure zero.) Show that $f$ is integrable iff $g$ is, and that if they are both integrable, then $\int_A f = \int_A g$.

**Solution**

The closure of $B$ is compact and has measure zero so it has content zero (pf: any cover of $\overline{B}$ by rectangles has a finite subcover). Therefore $B$ itself has content zero.

We'll show that the the upper integrals of $f$ and $g$ must be the same. I.e. that $\mathbf{U}(f) = \inf\{U(f,P)\} = \inf\{U(g,P)\} = \mathbf{U}(g)$. Equality of the lower integrals $\mathbf{L}(f) = \mathbf{L}(g)$ holds by a similar argument. It follows immediately that $f$ is integrable iff $g$ is, and that if the $f$ and $g$ are integrable then then $\int_A f = \int_A g$.

A few observations will carry us through:

1. Convince yourself that if $B \subset A$ has content zero and $P$ is a partition of $A$, then given any $\epsilon > 0$ there exists a refinement $P'$ of $P$ st

$$\sum_{P_i \in P' st P_i \cap B \neq \emptyset} v(P_i) < \epsilon.$$

2. Given a partition $P$ of $A$ and $\epsilon > 0$ there exists a refinement $P'$ of $P$ st

$$|U(f,P') - U(g,P')| = |\sum_{P'} (M_{P_i}(f) - M_{P_i}(g)) v(P_i)| < \epsilon.$$

Pf: Note that if $B \cap P_i = \emptyset$ for some rectangle $P_i$ in a partition of $A$, then $M_{P_i}(f) = M_{P_i}(g)$ and $m_{P_i}(f) = m_{P_i}(g)$. Hence, stating the contrapositive, $|M_{P_i}(f) - M_{P_i}(g)| \neq 0$ or $|m_{P_i}(f) - m_{P_i}(g)| \neq 0$ implies there exists $x \in B \cap P_i$. So the collection of rectangles in a partition $P$ where the maxima or minima of $f$ and $g$ disagree is a subset of the collection $\{P_i \in P | P_i \cap B \neq \emptyset\}$. Hence

$$\begin{aligned} & |U(f,P) - U(g,P)| \\ \leq & \sum_P |(M_{P_i}(f) - M_{P_i}(g))| \cdot v(P_i) \\ \leq & (|M_A(f)| + |M_A(g)|) \sum_{P_i \in P st P_i \cap B \neq \emptyset} v(P_i) \end{aligned}$$

From 1, this last quantity can be made as small as we want with some refinement $P'$ of $P$, hence the claim follows.

We proceed now to show $B$ has content zero implies $\mathbf{U}(f) = \mathbf{U}(g)$. We claim that $\mathbf{U}(g)$ is a lower bound for $\{U(f, P)\}$. Suppose there exists $P$ st $\mathbf{U}(g) - U(f, P) = \epsilon > 0$. Then by 2 there's a refinement $P'$ of $P$ st $|U(f, P') - U(g, P')| < \epsilon$. Recalling that $P'$ refines $P$ implies $U(f, P') \leq U(f, P)$, we see that $U(g, P') < \mathbf{U}(g)$, a contradiction since $\mathbf{U}(g)$ is the infimum of such sums. Hence $\mathbf{U}(g)$ is a lower bound for $\{U(f, P)\}$.

Now,

$$|U(f, P) - \mathbf{U}(g)| \leq |U(f, P) - U(g, P)| + |U(g, P) - \mathbf{U}(g)|$$

and by 2 and the definition of $\mathbf{U}(g)$ we can make both summands on the right as small as we want, and therefore the quantity on the left can be made as small as we want with proper choice of $P$. Hence, $\mathbf{U}(g)$ is not just a lower bound but also the *greatest* lower bound of $\{U(f, P)\}$, so $\mathbf{U}(g) = \mathbf{U}(f)$. As mentioned above, we can get $\mathbf{L}(f) = \mathbf{L}(g)$ by similar arguments. It follows that $\mathbf{U}(f) = \mathbf{L}(f)$ iff $\mathbf{U}(g) = \mathbf{L}(g)$. That is, $f$ is integrable iff $g$ is integrable. Furthermore equality of the integrals of $f$ and $g$ (provided they are integrable!) follows immediately.