MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (part 1) Solutions
March 21, 2005

1. True or False (22 points, 2 each)

T or **F**   Every set in $\mathbb{R}^n$ is either open or closed (or both).
False. A set such as $(0, 1] \subset \mathbb{R}$ is neither.

T or **F**   Every subset of a compact set in $\mathbb{R}^n$ is compact.
False. Every *closed* subset of a compact set is compact, but compact sets can have open (and hence non-compact) subsets.

T or **F**   The intersection of every countable nested collection of non-empty bounded closed sets in $\mathbb{R}^n$ is non-empty.
True. The Cantor Intersection Theorem states that this is true. In fact, it is true even if the countable condition is removed, but the proof is harder.

T or **F**   For any set $A \subset \mathbb{R}^n$, we have $\overline{\overline{A}} = \overline{A}$.
True. This was an exercise.

T or **F**   If $A \subset \mathbb{R}^n$ is discrete, then A has no accumulation points.
False. Consider the example $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$. Then $S \cap (\frac{1}{n+1}, \frac{1}{n-1}) = \{\frac{1}{n}\}$ so that S is discrete, but the real number 0 is an accumulation point.

T or **F**   Every non-empty set $A \subset \mathbb{R}^n$ has a non-empty boundary.
False. Consider the example $A = \mathbb{R}^n$ itself, which has empty boundary.

T or **F**   Let $A \in M_n(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$ be fixed. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, then $Jf = A$.

True. This is a fact from class.

For the next two questions, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable:

T or **F**   If $A \subset \mathbb{R}^n$ is any set and $df_{\mathbf{a}} = 0$ for every $\mathbf{a} \in A$, then f is constant on A.

False. This is true if A is connected, but consider $A = (0,1) \cup (2,3)$ with $f(x) = 0$ if $x \in (0,1)$ and $f(x) = 1$ if $x \in (2,3)$.

T or **F**   If $B \subset \mathbb{R}^n$ is compact, then $f(B) \subset \mathbb{R}^m$ is compact.

True. This follows from the homework because if f is differentiable, then f is also continuous.

For the last two questions, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous.

T or **F**   If $A \subset \mathbb{R}^n$, then $f^{-1}(f(A)) = A$.

False. Consider $A = [0,2] \subset \mathbb{R}$ and $f(x) = x^2$. Then $f(A) = [0,4]$ and $f^{-1}(f(A)) = [-4,4]$.

T or **F**   If $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$, then $\{f(\mathbf{x}_n)\}$ converges to $f(\mathbf{x})$.

True. This is one of the many possible definitions of continuity.

2. True or False with Explanation (12 points, 4 each)

For each of the following statements, identify whether the statement is True of False (2 points). If the statement is True, give a *short* explanation (e. g. cite a theorem from class, give the basic idea of the proof, etc.), and if the statement is False, provide a counter-example (2 points).

(a) **If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^n$ is open, then $f(A)$ is open.**

False. A similar but true statement is: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $A \subset \mathbb{R}^m$ is open, then $f^{-1}(A)$ is open. For an example of why the original statement is false, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ for every $x \in \mathbb{R}$. If we take the open interval $A = (a, b)$, then $f(A) = \{0\}$, which is a single point, and hence is closed.

(b) **Every bounded infinite set in $\mathbb{R}^n$ has an accumulation point.**

True. This is the Bolzano-Weierstrass Theorem.

(c) **If $B \subset \mathbb{R}^n$ and $\{S_\alpha\}_{\alpha \in I}$ is an open cover of B, then this cover has a finite subcover.**

False. In order for this statement to be true, we would have to add the condition that B is compact. (A similar sounding true statement is the Cantor Intersection Theorem, which states: If $B \subset \mathbb{R}^n$ and $\{S_\alpha\}_{\alpha \in I}$ is an open cover of B, then this cover has a *countable* subcover.) For an example of why the original statement is false, consider the set $B = (0, 1) \subset \mathbb{R}$, and consider the cover $\{S_n\}_{n=2}^{\infty}$ where $S_n = (\frac{1}{n}, 1)$. For another example, consider $B = \mathbb{R}^n$ and let the cover consist of all balls of radius one.

3. Continuity and Differentiability (12 points, 3 per column)

(a) **Consider each of the following four conditions that could describe a function $f : \mathbb{R}^2 \to \mathbb{R}$:**

  - **differentiable at** $(0,0)$
  - **continuous at** $(0,0)$
  - **continuously differentiable at** $(0,0)$
  - **partial derivatives exist at** $(0,0)$

  In the first column of the table below, list these conditions from _weakest_ to _strongest_.

  We have theorems from class that state:

  - If f is continuously differentiable at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.
  - If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.
  - If f is differentiable at $\mathbf{a}$, then all of f's directional derivatives exist at $\mathbf{a}$.

  There is no relationship between a function being continuous at a point and having all of its directional derivatives exist at that point. For example, $f(\mathbf{x}) = ||\mathbf{x}||$ is continuous at $(0,0)$, but its partial dervatives are undefined there, and $g(x, y) = \frac{x^2 y}{x^4 + y^2}$ has directional derivative 0 in any direction at $(0,0)$, but is not continuous there–note that at any point of the form (a, a^2), we have $g(a, a^2) = \frac{1}{2}$ and that points of this form get arbitrarily close to the origin.

(b) **Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:**

$$f(x, y) = \frac{x^n}{x^2 + y^2} \ \textbf{ for } (x, y) \neq (0,0) \ \textbf{ and } \ f(0,0) = 0$$

  where n is a positive integer.

  For each integer value of n from 2 to 4 and for each condition A through D, indicate in the table below whether f satisfies the corresponding condition with a "Y" for yes and an "N" for no.

- For $n = 2$, $\lim_{x \to 0} f(x, 0) = 1$ and $\lim_{y \to 0} f(0, y) = 0$, so f is not continuous at $(0, 0)$, and the directional derivative $D_{\mathbf{e}_1} f(0, 0)$ is undefined. Hence f fails all four conditions.

- For $n = 3$, the Squeeze Theorem implies that f is continuous, and if we recall homework problem #5.2, we see that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$. On that problem, we showed that any *differentiable* function satisfying this homogeneous relation and the relation $f(\mathbf{v}) = \nabla f(\mathbf{0}) \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ must be *linear*, but that since f is not linear, we conclude that it possesses directional derivatives in all directions at $(0, 0)$, but that f is *not* differentiable at $(0, 0)$.

- For $n = 4$, all of the second-order partials of f exist, and hence f satisfies all four of our conditions. The basic idea is that as n increases, we "smoothing out" f.

Thus the chart should have read something like this, with the possibility of switching rows A and B:

Condition	$n = 2$	$n = 3$	$n = 4$
$A.f$ is continuous at $(0, 0)$	N	Y	Y
$B.$all of f's directional derivatives exist at $(0, 0)$	N	Y	Y
$C.f$ is differentiable at $(0, 0)$	N	N	Y
$D.f$ is continuously differentiable at $(0, 0)$	N	N	Y

4. **The Topology of $\mathbb{R}^n$** (16 points, 3/3/5/5)

Consider the following definitions:

> **Definition.** A set $S \subset \mathbb{R}^n$ is said to be *thin* if it is closed and has empty interior.

> **Definition.** A set $T \subset \mathbb{R}^n$ is said to be *dense* in $\mathbb{R}^n$ if, given any $\mathbf{x} \in \mathbb{R}^n$ and any $\varepsilon > 0$, the ball of radius ε around $\mathbf{x}$ intersects T non-trivially, that is, $B_\varepsilon(\mathbf{x}) \cap T \neq \varnothing$.

(a) **Give an example of a (non-empty) thin set in $\mathbb{R}^2$.**

Several examples: $A = \{(0,0)\}$, $B = \{(x,y) \mid x^2 + y^2 = 1\}$, or $C = \{(x,y) \mid y = 0\}$.

(b) **Give an example of a (non-trivial) dense set in $\mathbb{R}^2$.**

Several examples: $A = \mathbb{R}^n \setminus \{(0,0)\}$, $B = \{(x,y) \mid x^2 + y^2 \neq 1\}$, or $C = \{(x,y) \mid x, y \in \mathbb{Q}\}$. Note that $\mathbb{R}^n$ itself would be considered a trivial example.

(c) **Show that if $B \subset \mathbb{R}^n$ is thin, then B^c is open and dense.**

If B is thin, then by definition, it is closed, and therefore B^c is open (fact about open and closed sets). Now we show that B^c is dense by considering any point $\mathbf{x} \in \mathbb{R}^n$. If $\mathbf{x} \in B^c$, then for any $\varepsilon > 0$, we have $B_\varepsilon(\mathbf{x}) \cap B^c \supset \{\mathbf{x}\} \neq \varnothing$, and so the density condition is satisfied. Conversely, if $\mathbf{x} \notin B^c$, then $\mathbf{x} \in B$. If there were an $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \cap B^c = \varnothing$, then $B_\varepsilon(\mathbf{x}) \subset B$ and $\mathbf{x}$ would be an interior point for B, which cannot happen since B is thin. Therefore $B_\varepsilon(\mathbf{x}) \cap B^c \neq \varnothing$, and so the density is once again satisfied.

(d) **Show that if $A \subset \mathbb{R}^n$ is open, then ∂A is thin.**

Note that $\partial A = \overline{A} \cap \overline{A^c}$ (by homework problem # 2.3). This is the intersection of two closed sets, which is also closed (by homework problem # 1.12). To show that ∂A has empty interior, consider any $\mathbf{x} \in \partial A$. If there is some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset \partial A$, then $B_\varepsilon(\mathbf{x}) \cap A = \varnothing$ (since $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap A$ (because A is open and hence $A^\circ = A$) and hence $\partial A \subset A^c$). But this contradicts the definition of boundary which says that for any $\varepsilon > 0$, we must have $B_\varepsilon(\mathbf{x}) \cap A \neq \varnothing$. Hence $\mathbf{x}$ is not an interior point.

5. Some Computations (15 points, 3 each)

Consider the following functions:

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad \textbf{given by} \qquad f(u, v) = \tfrac{1}{4}u^2 \cdot (v - 2)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \textbf{given by} \qquad g(x, y) = (x + y^3, y - e^x)$$

$$h : \mathbb{R}^2 \to \mathbb{R} \qquad \textbf{given by} \qquad h(x, y) = f(g(x, y))$$

If $\mathbf{a} = (0, 2)$, $\mathbf{b} = (1, 5)$, and $\mathbf{v} = (1, 1)$, find/compute the following:

(a) $\nabla f = \left(\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v} \right) = \left(\dfrac{1}{2}u(v - 2), \dfrac{1}{4}u^2 \right)$

(b) Easy way: $D_{\mathbf{v}} f(\mathbf{b}) = \nabla f(\mathbf{b}) \cdot \mathbf{v} = (\tfrac{3}{2}, \tfrac{1}{4}) \cdot (1, 1) = \tfrac{7}{4}$

Hard way: $D_{\mathbf{v}} f(\mathbf{b}) = \lim\limits_{h \to 0} \dfrac{f(\mathbf{b} + h\mathbf{v}) - f(\mathbf{b})}{h} = $ etc.

(c) The equation of the plane tangent to f at $\mathbf{b}$ is:

$$\begin{aligned}
T(\mathbf{x}) &= f(\mathbf{b}) + \nabla f(\mathbf{b}) \cdot (\mathbf{x} - \mathbf{b}) \\
T(u, v) &= f(1, 5) + (\tfrac{3}{2}, \tfrac{1}{4}) \cdot (u - 1, v - 5) \\
&= \tfrac{3}{4} + \tfrac{3}{2}(u - 1) + \tfrac{1}{4}(v - 5)
\end{aligned}$$

(d) $Jg = \begin{bmatrix} 1 & 3y^2 \\ -e^x & 1 \end{bmatrix}$ and so $Jg(\mathbf{a}) = \begin{bmatrix} 1 & 12 \\ -1 & 1 \end{bmatrix}$

(e) By the Chain Rule,

$$\nabla h(\mathbf{a}) = \nabla f(g(\mathbf{a})) Jg(\mathbf{a}) = \begin{bmatrix} -4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 12 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -20 & -32 \end{bmatrix}$$