Last Name: ______________________________

First Name: ______________________________

MATH 23b, SPRING 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam
May 16, 2003

Directions:

You have three hours for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

Problem	Points	points by part	Score
1	20	1 each	
2	9	3 each	
3	8	4/2/2	
4	16	8 each	
5	9	4/3/2	
6	9	2/3/4	
7	9	3 each	
8	5		
Total	76	76	

1. **True or False**

T or **F** If $A = \{x_i\}_{i=1}^{\infty}$ is a bounded subset of $\mathbb{R}^n$, then A has an accumulation point.

T or **F** If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, then JL is linear.

T or **F** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $JL = L$.

T or **F** The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is constant if and only if $Jf = 0$.

T or **F** The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if and only if all of its directional derivatives exist and are continuous at $\mathbf{a}$.

T or **F** There exists a twice-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y}$.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable, then f is continuously differentiable.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is locally invertible at $\mathbf{a} \in \mathbb{R}^n$, then $\det Jf(\mathbf{a}) \neq 0$.

T or **F** The point $(0, 0, 0)$ is a saddle point of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + 2xy + y^2 - z^2$.

T or **F** If $A \subset \mathbb{R}^n$ has measure zero, then A is countable.

T or **F** If $A \subset \mathbb{R}^n$ has content zero, then A has measure zero.

T or **F** The Cantor set (defined in the usual way as a subset of $[0, 1]$) has content zero.

T or **F** Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be bounded. If P is a partition of A and P' is a refinement of P, then $L(f, P) \leq L(f, P')$.

T or **F** Let $A \subset \mathbb{R}^n$ be bounded, and suppose A has (defined) non-zero content. Suppose ∂A has measure zero. If $f : A \longrightarrow \mathbb{R}^n$ is bounded, then f is integrable on A.

T or **F** If $A \subset \mathbb{R}^n$ is a closed rectangle and $f : A \to \mathbb{R}$ is bounded, then $\mathcal{L}\int_A f$, the lower-integral of f, exists.

T or **F** Let $A \subset \mathbb{R}^n$ be an open ball, and let $f : A \longrightarrow \mathbb{R}$ be in the class C^2. Then ∇f is conservative.

T or **F** Let $U \subset \mathbb{R}^n$ be an open ball, and let $\mathbf{x}_1, \mathbf{x}_2 \in U$. Let C be a piece-wise smooth curve (in U) from $\mathbf{x}_1$ to $\mathbf{x}_2$, and let $\gamma_1 : [a,b] \to C$ and $\gamma_2 : [c,d] \to C$ be two parametrizations of C. Then $\int_a^b F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt = \int_c^d F(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$.

T or **F** Let $U \subset \mathbb{R}^n$ be an open ball, and let $\mathbf{x}_1, \mathbf{x}_2 \in U$. If C_1 and C_2 are two piece-wise smooth curves (in U) from $\mathbf{x}_1$ to $\mathbf{x}_2$, then $\int_{C_1} F = \int_{C_2} F$.

T or **F** If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is class C^1, then $\mathrm{curl}(F) = 0$.

T or **F** If $A, B \subset \mathbb{R}$ are closed intervals and $f : A \times B \to \mathbb{R}$ is bounded, then

$$\int_B \int_A f(x,y)\, dx\, dy = \int_A \int_B f(x,y)\, dy\, dx$$

provided that at least one of these iterated integrals exists.

2. **Examples**

Provide examples (with a line or two of justification) of each of the following:

(a) A set $A \subset \mathbb{R}^n$ with undefined content.
(You may choose both n and A.)

(b) A closed rectangle $A \subset \mathbb{R}^n$ and a bounded function $f : A \to \mathbb{R}$ whose upper and lower integrals are unequal.
(You may choose n as well as A and f.)

(c) An open cover of the interval $(0, 1)$ that has no finite subcover.

3. **Quadratic Forms**

(a) State the Spectral Theorem for Real Symmetric Matrices.

(b) Diagonalize the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$q(x, y, z) = 2x^2 + 5y^2 + 2xz + 2z^2.$$

(c) Use part (b) to decide whether q is positive, positive-definite, negative, negative-definite, or none of these.

4. **Integration Theory**

(a) Let $f : [a, b] \to \mathbb{R}$ be an *increasing* function, that is, if $x, y \in [a, b]$ and $x < y$, then $f(x) < f(y)$. Show that f is integrable on $[a, b]$.

(b) Let $A \subset \mathbb{R}^n$ be compact, and let $f : A \to \mathbb{R}$ be continuous and also non-negative, that is, $f(\mathbf{x}) \geq 0, \forall \mathbf{x} \in A$. Show that if there is some $\mathbf{a} \in A$ with $f(\mathbf{a}) > 0$, then $\int_A f > 0$.

5. **Double Integrals**

Let $D = \overline{B_1(\mathbf{0})}$ be the closed unit disk in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{1}{(x^2 + y^2)^n}$$

where $n > 0$. (Note that f is unbounded on D.)

(a) Construct an approximating sequence $\{A_i\}_{i=1}^{\infty}$ for D so that f is integrable on each A_i.

(b) Evaluate $\displaystyle\int_{A_i} f$.

(c) For what values of n does the integral $\displaystyle\int_D f$ converge?

6. **Line Integrals**

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field $F(x, y) = (x^2y + 4x, kx^3 + 1)$.

(a) Find the unique value of the constant k that makes F conservative.

(b) Find a potential function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = F$ for the value of k you found in part (a).

(c) Compute the line integral $\int_C F$, where C is the straight line segment from $(0, 3)$ to $(2, 7)$ and k is the value from part (a).

7. **Green's Theorem**

(a) State Green's Theorem (including all hypotheses!) for a vector field $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $F(x, y) = (P(x, y), Q(x, y))$.

(b) Explain the impact of Green's Theorem when F is conservative. (*i.e.* How does this relate to the Fundamental Theorem of Line Integrals?)

(c) Explain why Green's Theorem does not apply to the vector field

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

on a region containing the origin.

8. **An Improper Integral**

Please note that this problem is not one of the preview problems, though it is somewhat related.

Let $q : \mathbb{R}^n \to \mathbb{R}$ be a positive-definite quadratic form.

Compute $\displaystyle\int_{\mathbb{R}^n} e^{-q(\mathbf{x})}\, d\mathbf{x}$.

Hint: Recall that $\displaystyle\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$.

Remark: Not every step of your reasoning must be completely justified, but you should try to indicate the steps you are doing and why they are reasonable.