Last Name: ______________________________

First Name: ______________________________

MATH 23b, SPRING 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam
May 28, 2004

Directions:

You have three hours for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. (There are blank pages included after problems 2 and 6 for extra work.) Show all work!

Problem	Points	points by part	Score
1	40	2 each	
2	20	10 each	
3	20	4/4/12	
4	10		
5	15	5 each	
6	20	10 each	
Total	125	125	

1. **True or False**

T or **F** All bounded sets in $\mathbb{R}^n$ are either open or compact.

T or **F** Discrete sets in $\mathbb{R}^n$ have no accumulation points.

T or **F** If $A \subset \mathbb{R}^n$ is open, then every $\mathbf{x} \in A$ is an interior point of A.

T or **F** If $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the usual Euclidean topology, then $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) \neq 0\}$ is an open set.

T or **F** If $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the usual Euclidean topology, then $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) = 1\}$ is a compact set.

T or **F** Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let $\mathbf{b} \in \mathbb{R}^m$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is given by $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, then $Jf = A$.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$, then all of its directional derivatives exist at $\mathbf{a}$.

T or **F** If $f : \mathbb{R}^2 \to \mathbb{R}$ is in the class C^3, then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

T or **F** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be in the class C^3. If $\nabla f(\mathbf{a}) = \mathbf{0}$ and $\nabla^2 f(\mathbf{a}) > 0$, then f has a local minimum at $\mathbf{a}$.

T or **F** According to the Inverse Function Theorem, the function $f(x, y) = (xy, x^2 - y^2)$ is locally invertible at every point of its domain.

T or **F** The set $A = \{(x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ has measure zero.

T or **F** The set of all real numbers with no 8's in their decimal expansions has measure zero (considered as a subset of $\mathbb{R}$).

T or **F** Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be bounded. If P is a partition of A and P' is a refinement of P, then $L(f, P) < L(f, P')$.

T or **F** Let $f : \mathbb{R}^n \to \mathbb{R}$. If $o(f, \mathbf{a}) = 0$, then f is continuous at $\mathbf{a}$.

T or **F** Let $f : \mathbb{R}^n \to \mathbb{R}$. If $o(f, \mathbf{a}) > 0$, then f is discontinuous at $\mathbf{a}$.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, then f is integrable on any compact set $A \subset \mathbb{R}^n$.

T or **F** If $f : [a, b] \to \mathbb{R}$ is increasing, then f is integrable on $[a, b]$.

T or **F** If $A \subset \mathbb{R}^n$ is a closed rectangle and $f : A \to \mathbb{R}$ is bounded, then $\mathcal{L} \int_A f$, the lower-integral of f, exists.

T or **F** Let $U \subset \mathbb{R}^n$ be an open ball, and let $\mathbf{x}_1, \mathbf{x}_2 \in U$. Let C be a piece-wise smooth curve (in U) from $\mathbf{x}_1$ to $\mathbf{x}_2$, and let $\gamma_1 : [a, b] \to C$ and $\gamma_2 : [c, d] \to C$ be two parametrizations of C. Then $\int_a^b F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt = \int_c^d F(\gamma_2(t)) \cdot \gamma_2'(t)\, dt$.

T or **F** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field with coordinate functions $F = (P, Q)$. If P and Q are both continuously differentiable, then F is conservative.

2. **Optimization Problems**

(a) (Theoretical) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and let $S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = 0 \text{ and } \nabla g(\mathbf{x}) \neq \mathbf{0}\}$.

i. Use the Implicit Function Theorem to justify that every point on S has a neighborhood in which one of the variables may be written as a function of the other $n-1$ variables. (In particular, this implies that S is an $(n-1)$-manifold.)

ii. Let $\mathbf{p} \in \mathbb{R}^n$ be a fixed point *not* on S, and let $\mathbf{q}$ be the point on S closest to $\mathbf{p}$. (We are assuming that such a $\mathbf{q}$ exists.) Prove that the vector $\mathbf{p} - \mathbf{q}$ is orthogonal to any vector tangent to S at $\mathbf{q}$. *(Hint: Consider the function $f(\mathbf{x}) = ||\mathbf{p} - \mathbf{x}||^2$.)*

(b) (Practical) Use the method of Lagrange multipliers to find the dimensions of the largest rectangle (in terms of area) that can be inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. **Integration Theory**

(a) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be bounded. Define what it means for f to be *integrable* on A.

(b) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \to \mathbb{R}$ be bounded. State the theorem that most completely characterizes whether or not f is integrable on A.

(c) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and suppose that $f : A \to \mathbb{R}$ is both bounded and integrable. Show that if $f(\mathbf{x}) > 0, \forall \mathbf{x} \in A$, then $\int_A f > 0$.

4. **A Double Integral**

Let $D = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ and } x, y \geq 0\}$ be the portion of the closed unit disk in the first quadrant of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & , \text{ if } (x, y) \neq (0, 0) \\ 0 & , \text{ if } (x, y) = (0, 0) \end{cases}$$

Justify that f is integrable on D, and compute $\displaystyle\int_D f$.

5. **Line Integrals**

(a) Let $F(x, y, z) = (y, -x, 2z)$, and let C be the straight line segment from $\mathbf{a} = (1, 2, 3)$ to $\mathbf{b} = (2, 4, 7)$.

Parametrize C and compute $\int_C F$ using the definition.

(b) State the Fundamental Theorem of Line Integrals.

(c) Could you have used the Fundamental Theorem of Line Integrals from part (b) to evaluate the line integral from part (a)? Why or why not?

6. **Stokes' Theorem**

Let S be a parametrized surface in $\mathbb{R}^3$ with oriented boundary C. That is, suppose there exists some $D \subset \mathbb{R}^2$ that is open, connected, and simply-connected, with a boundary ∂D that is a piece-wise smooth, positively-oriented curve, and a function $\varphi : \overline{D} \to S$ that is continuous on $\overline{D}$ and continuously differentiable (except possibly on a set of measure zero) and bijective on D. By convention, denote the independent variables as u and v.

Let $F : U \to \mathbb{R}^3$ be a vector field that is class C^1 and represented by the components $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$, where $U \subset \mathbb{R}^3$ is some open set containing S.

Theorem. (Stokes' Theorem): $\displaystyle\int\int_S \text{curl } F = \int_C F$

Note that the right-hand side is a standard line integral around the piece-wise smooth, positively-oriented (with respect to a right-hand coordinate system) curve C. The left-hand side is a flux integral, which, by definition, is evaluated as follows. If $G : \mathbb{R}^3 \to \mathbb{R}^3$ is any continuous vector field, then, with the parametrization above,

$$\begin{aligned}\int\int_S G &= \int\int_S G \cdot \mathbf{n}\, dS \\ &= \int\int_D G(\varphi(u, v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) du\, dv,\end{aligned}$$

where $\mathbf{n}$ is the normal vector to the surface S, and u and v are the two variables parametrizing D as a subset of $\mathbb{R}^2$. It can be shown (though you need not do so on this exam) that the value of the flux integral is independent of the parametrization chosen.

Stokes' Theorem, then, applies to the special case when the vector field in the flux integral is itself the curl of another vector field. With F as above, recall that we define:

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

(a) By direct computation, prove Stokes' Theorem in the case that S is a rectangle parallel to the yz-plane (and whose edges are parallel to the y and z axes), that is, one of the form

$$S = \{(x, y, z) \mid x = a,\ b \leq y \leq b',\ c \leq z \leq c'\},$$

for some constants $a, b, b', c, c' \in \mathbb{R}$.

(You may not assume Green's Theorem!)

(b) Verify Stokes' Theorem for the vector field $F(x, y, z) = (z, x, y)$ on the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$, the upper-hemisphere of the sphere of radius one, centered at the origin, with boundary $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z = 0\}$.