MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Questions for the Final

Directions: Here are five questions to help you prepare for the final, which will take place at 9:15 A.M. on Tuesday, May 24 in Harvard Hall 104. At least one of the final list of questions will be on the final, either verbatim or in some small variation.

You are encouraged to work with other members of the class on these problems or to discuss them with the instructor, but in the interests of fairness, please do not ask the course assistants about these problems.

1. Let $f : [a, b] \to \mathbb{R}$ be an *increasing* function, that is, if $x, y \in [a, b]$ and $x < y$, then $f(x) < f(y)$. Show that f is integrable on $[a, b]$.

2. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and suppose $f : A \to \mathbb{R}$ is integrable. Show that if $f(x) > 0, \forall x \in A$, then $\int_A f > 0$.

3. Let $A \subset \mathbb{R}^n$ be a closed rectangle. Consider the function $f : A \to \mathbb{R}$, and consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ where $f_n : A \to \mathbb{R}$ for each $n \in \mathbb{N}$. We say that:

   - $\{f_n\}$ *converges pointwise* to the function f provided that, for each $\mathbf{x} \in A$, the sequence of real numbers $\{f_n(\mathbf{x})\}_{n=1}^{\infty}$ converges to $f(\mathbf{x})$.
   - $\{f_n\}$ *converges uniformly* to the function f provided that, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$ for every $n > N$ and for every $\mathbf{x} \in A$.

   (a) Show that if, for each $n \in \mathbb{N}$, the function $f_n : A \to \mathbb{R}$ is continuous, and the sequence $\{f_n\}$ converges uniformly to $f : A \to \mathbb{R}$, then f is continuous.

   (b) Let $A = [0, 1] \subset \mathbb{R}$, and for each $n \in \mathbb{N}$, consider the function $f : A \to \mathbb{R}$ defined by $f_n(x) = 1/(x + 1)^n$.

      i. Find the function $f : A \to \mathbb{R}$ to which the sequence $\{f_n\}$ converges pointwise.

      ii. Does $\{f_n\}$ converge to f uniformly? Explain.

4. We say that a set $X \subset \mathbb{R}^n$ is *disconnected* if there exist two open sets $A, B \subset \mathbb{R}^n$ such that:

   - $X \subset A \cup B$ (that is, A and B cover X)
   - $A \cap B = \emptyset$ (that is, A and B are disjoint)
   - $A \cap X \neq \emptyset$ and $B \cap X \neq \emptyset$
     (that is, both A and B contain part of X)

   We say that a set $X \subset \mathbb{R}^n$ is *connected* if it is not disconnected.

   (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $X \subset \mathbb{R}^n$ is connected. Show that $f(X)$ is connected.

   (b) Use the fact that an interval $[a, b] \subset \mathbb{R}$ is connected to prove the:

   Intermediate Value Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and y is any real number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$.

   (c) Use the contrapositive of the statement in part (a) to show that $O_n(\mathbb{R})$ is disconnected as a subset of $\mathbb{R}^{n^2}$. (Recall that

   $$O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^t A = I\}.$$

   See homework set #3 for more details.)

5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Recall (see homework #5.12 and 5.13) that we defined the *Laplacian* of f as follows:

   $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

   We defined a function to be *harmonic* if $\nabla^2 f = 0$.

   Now we define f to be *subharmonic* if $\nabla^2 f > 0$.

   (a) Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the closed unit disk. Show that if $f : D \longrightarrow \mathbb{R}$ is subharmonic, then f does not have a maximum on the interior of D.

   (b) Give an example of a function $f : D \to \mathbb{R}$ that is subharmonic on the closed unit disk, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.