MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 1 (Final Version)
Due: October 1, 2004

Starred problems are for your enrichment, but they need not be written up and turned in to be graded.

Beginning with this problem set, we ask that you turn in four separate sheets (or sets of sheets) labelled A through D, so that the CA's may grade them in parallel. The individual problems are each labelled with one of A through D below.

1. Reading: Section 3.1 from Schneider and Barker and Section 1.1 from Edwards.
2. (A) Use mathematical induction to show that, for any positive integer k, we have $\sum_{i=1}^{k}(2i-1) = k^2$.

   (Whether or not you have seen this type of problem before, do it as an exercise in using the formalism we developed when discussing Peano's Postulates.)
3. (B) Verify that multiplication is well-defined for integers, as defined by equivalence classes of ordered pairs of natural numbers.
4. (*) Use mathematical induction to prove the following:

   If the Euclidean plane is cut by a finite number of lines, the connected regions created by those lines may be colored in black and white in such a way that any two such regions that share a line segment as part of their boundaries have different colors.

   Bonus: Given n lines, what is the maximum number of regions the plane may be cut into?
5. (C) In this exercise, you will use the integers and their properties to construct the rational numbers. For purposes of this exercise, you may use any fact that we have proved (or should have proved!) about the integers, and you should denote an integer by a single variable instead of an equivalence class, i.e. use a instead of $[(a, b)]$.

   (a) Make a good definition of the rational numbers as equivalence classes of pairs of integers.

(b) For two rational numbers a and b (as you defined them), define addition $(a+b)$, multiplication $(a \cdot b)$, and less than $(a < b)$. *(For purposes of this problem, you do not need to check that these are well-defined, but perhaps you might include some motivation for your definition.)*

(c) Prove the existence of additive inverses. *(Hint: To do this, you will need first to name the additive identity correctly.)*

(d) Prove that the positive rational numbers are closed under addition. *(Hint: You may assume that by "positive", we mean the elements that are greater than 0.)*

6. (deferred) In class, we constructed the ring $\mathbb{Z}/2\mathbb{Z}$, which consisted of two equivalence classes of integers. In this exercise, you will construct $\mathbb{Z}/3\mathbb{Z}$.

   (a) Identify the ring $3\mathbb{Z}$.

   (b) Define $\mathbb{Z}/3\mathbb{Z}$ using the appropriate equivalence relation.

   (c) Identify all of the equivlance classes in $\mathbb{Z}/3\mathbb{Z}$.

   (d) Construct tables for addition and multiplication in $\mathbb{Z}/3\mathbb{Z}$ using the definitions: $[a]+[b]=[a+b]$ and $[a]\cdot[b]=[a\cdot b]$

   (e) Is $\mathbb{Z}/3\mathbb{Z}$ a field?

7. (*) Find a necessary and sufficient condition on the natural number n such that $\mathbb{Z}/n\mathbb{Z}$ is a field.

8. (D) Show that additive inverses in fields are unique. That is, show that given $a \in F$, there exists a *unique* element $b \in F$ such that $a+b=0$.

9. (*)

   Let $x>0$ be a real number. Show that there is an integer k such that x may be represented in the form

$$x=\sum_{i=k}^{\infty} a_i \cdot 10^{-i} = a_k \cdot 10^{-k} + a_{k+1} \cdot 10^{-k-1} + \cdots + a_{-1} \cdot 10^{1} + a_0 \cdot 10^{0} + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots$$

   where $a_i \in \{0,1,2,\ldots,9\}$ for every $i \geq k$. Show that this representation is unique, except in the case where there exists some $n \in \mathbb{N}$ such that $10^n \cdot x \in \mathbb{N}$.

   (Hint: Use the Well-Ordering Principle and perhaps the Division Algorithm.)