MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 10
Due: April 29, 2005

Homework Assignment #10 (Final Version)

1. Read Edwards, Sections 4.1–4.3.

2. (*) Show that if a compact set $A \subset \mathbb{R}^n$ has measure zero, then it has content zero.

3. (A) Let $A = \{x \in [0,1] \mid \text{the decimal expansion of } x \text{ has no 8's}\}$.
   Let $B = \{n \in \mathbb{N} \mid \text{the decimal expansion of } n \text{ has no 8's}\}$.

   (a) Show that $A$ has content zero.

   (b) Decide whether the infinite series $\displaystyle\sum_{n \in B} \frac{1}{n}$ converges or diverges.

   *(Hint: These two parts actually have very little to do with each other, except superficially.)*

4. (*) Find a bijection between the Cantor set and the closed interval $[0,1]$. *(Hint: Consider the ternary expansion of numbers in the Cantor set.)*

5. (B) For a function $f : [0,1] \longrightarrow \mathbb{R}$, let $A = \{x \in [0,1] | f \text{ is not differentiable at } x\}$. Find such an $f$ satisfying the following conditions:

   - $f$ is continuous.
   - $f(0) = 0$
   - $f(1) = 1$
   - $A$ has content zero.
   - If $x \notin A$, then $f'(x) = 0$. (You do *not* need to show that if $x \in A$, then $f$ is not differentiable at $x$, but you should convince yourself of this fact.)

   *(Hint: Use the Cantor set.)*

6. (* = a good practice problem) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be positive and continuous, and suppose that

$$\int_D f = \int_0^1 \left( \int_y^{\sqrt{2-y^2}} f(x,y)\,dx \right) dy.$$

   Sketch the region $D$ and interchange the order of integration.

7. (C) In class, we proved the following theorem:

   **Theorem.** Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \longrightarrow \mathbb{R}$ be bounded. Let $B = \{x \in A \mid f \text{ is discontinuous at } x\}$. Then $f$ is integrable if and only if $B$ has measure zero.

   In the proof, we considered the set:

$$B_\varepsilon = \{x \in A \mid o(f, x) \geq \varepsilon\}$$

   and claimed that it was closed (and hence compact since $A$ is bounded). Prove this.

8. (C) Let $f : A \longrightarrow \mathbb{R}$ and $g : A \longrightarrow \mathbb{R}$ be two bounded functions such that the closure of $B = \{x \in A \mid f(x) \neq g(x)\}$ is a set of measure zero. (This implies that $f$ and $g$ agree except on a set of measure zero.) Show that $f$ is integrable if and only if $g$ is, and that if they are both integrable, then $\displaystyle\int_A f = \int_A g$.

9. (D) In class, we considered the set $A = \mathbb{Q} \cap [0, 1]$ and showed that it has measure zero. In particular, we showed that it was countable, that is, we could write it as $A = \{a_0, a_1, a_2, a_3 \ldots\}$. Given an $\varepsilon > 0$, we then covered it with rectangles $I_i = [a_i - \frac{\varepsilon}{2^{i+2}},\ a_i + \frac{\varepsilon}{2^{i+2}}], \forall i \in \mathbb{N}$, so that

$$v\left(\bigcup_{i=0}^{\infty} I_i\right) \leq \sum_{i=0}^{\infty} v(I_i) = \varepsilon.$$

   Fix $\varepsilon = \frac{1}{2}$, and let $J_i = (a_i - \frac{1}{2^{i+3}}, a_i + \frac{1}{2^{i+3}})$ be the open rectangle equal to the interior of the corresponding $I_i$ defined above.

   Let $B = \bigcup_{i=2}^{\infty} J_i$. *(Note that we are purposely omitting the first two sets, which cover 0 and 1, respectively, so that each $J_i \subset [0, 1]$!)*

   (a) Show that $\partial B = [0, 1] \setminus B$.

   (b) Show that $\partial B$ does not have measure zero.

   (c) Let $\chi_B$ be the characteristic function of $B$. Show that $\chi_B$ is not integrable on $[0, 1]$.

   *(Note that although $B$ is a "reasonable" set in the sense that it is the union of a countable collection of open sets, it does not have a "reasonable" boundary, and so $\chi_B$ is not integrable.)*