MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Moral) Homework Assignment # 11
No due date

Homework Assignment # 11, or Problems to Think About

1. Read Edwards Sections 4.4–4.5 and 5.1–5.2.

2. (*) Let $S \subset \mathbb{R}^3$ be the (bounded) intersection of the two (unbounded) cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. Show that the volume of S is $\frac{16}{3}$.

   You can view an interactive feature with this object at:

   `http://www.math.umn.edu/~garrett/qy/Cylinders.html`

3. (*) Show that the volume of $B_1(0) \subset \mathbb{R}^4$ is $\frac{\pi^2}{2}$.

   (Hint: Use the fact that the volume of a ball in $\mathbb{R}^3$ of radius r is $\frac{4}{3}\pi r^3$.)

4. (*) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous, and let B_ε be the ball of radius ε centered at the point $\mathbf{x} \in \mathbb{R}^n$. Show that:

$$\lim_{\varepsilon \to 0} \frac{1}{v(B_\varepsilon)} \int_{B_\varepsilon} f = f(\mathbf{x}).$$

5. (*) Spherical coordinates in n dimensions (from Edwards' problem 5.19):

   (a) Three-dimensional Euclidean space can be represented via the standard spherical coordinates transformation:

$$(x, y, z) = T(\rho, \theta, \varphi) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi),$$

   where ρ is radius of the sphere on which (x, y, z) lies (the distance from the point to the origin), θ is the angle $(x, y, 0)$ makes with the x-axis (the longitude), and φ is the angle (x, y, z) makes with the z-axis (the latitude).

   Compute $|\det JT|$ as a function of ρ, θ, and φ.

   (b) More generally, n-dimensional spherical coordinates are given by the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given as:

$$\begin{aligned}
x_1 &= \rho \cos\varphi_1 \\
x_2 &= \rho \sin\varphi_1 \cos\varphi_2 \\
x_3 &= \rho \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
x_{n-1} &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\theta \\
x_n &= \rho \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\theta
\end{aligned}$$

Show by induction that

$$|\det JT| = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2}.$$

(c) Let $B^4 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \leq 1\}$ be the unit ball in $\mathbb{R}^4$, and let $f(x, y, z, w) = e^{(x^2+y^2+z^2+w^2)^2}$. Use the spherical coordinates change of variables to compute the integral $\int_{B^4} f$.

6. (*) In the following problem, we compute the volume of the n-dimensional unit ball, $B^n \subset \mathbb{R}^n$, in the even and odd cases, respectively, to be:

$$v(B^{2m}) = \frac{\pi^m}{m!} \qquad \text{and} \qquad v(B^{2m+1}) = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$$

(a) Read Edwards' problem 5.17 (p. 267) and problem 1.8 (p. 213) for one solution. Note that Edwards uses the notation I_n to denote the value of the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$.

(b) Do Edwards' problem 5.18 (p. 267):
Let $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$.
Let $Q = \{(x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \mid |x_i| \leq 1, \forall i\}$. Then $B^n \subset B^2 \times Q$.
Let $\varphi : B^2 \times Q \to \mathbb{R}$ be the characteristic function of B^n. Then

$$v(B^n) = \int_{B^2} \left(\int_Q \varphi(x_1, \ldots, x_n) \, dx_3 \ldots dx_n \right) dx_1 \, dx_2.$$

If $(x_1, x_2) \in B^2$ is fixed, then φ, considered as a function of the variables $x_3, \ldots, x_n$ is the characteristic function of B_r^{n-2}, the $(n-2)$-ball of radius $r = \sqrt{1 - x_1^2 - x_2^2}$. Hence

$$\int_Q \varphi(x_1, \ldots, x_n) \, dx_3 \ldots dx_n = (1 - x_1^2 - x_2^2)^{(n-2)/2} \cdot v(B^{n-2}).$$

Now, introduce polar coordinates on $\mathbb{R}^2$ to show that

$$\int_{B^2} (1 - x_1^2 - x_2^2)^{(n-2)/2} \, dx_1 dx_2 = \frac{2\pi}{n}.$$

Conclude that $v(B^n) = \frac{2\pi}{n} \cdot v(B^{n-2})$, and use induction and the base cases (that is, that $v(B^2) = \pi$ and $v(B^3) = \frac{4}{3}\pi$) to prove the given formulas for $v(B^n)$.

7. (*) Problem 5.7 from p. 264 of Edwards. Use the change of variables $u = x - y$ and $v = x + y$ to evaluate the integral $\int \int_D e^{(x-y)/(x+y)} \, dx \, dy$, where D is the region in $\mathbb{R}^2$ bounded by the axes $x = 0$ and $y = 0$ and the line $x + y = 1$.

8. (*) Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$.

   Let C_0 be the unit circle in $\mathbb{R}^2$ parametrized by the function $\gamma_0 : [0,1] \longrightarrow \mathbb{R}^2$, where $\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t)$.

   Let C_1 be the circle of radius 1 centered at $(1,1)$ in $\mathbb{R}^2$ parametrized by the function $\gamma_1 : [0,1] \longrightarrow \mathbb{R}^2$, where $\gamma_1(t) = (1 + \cos 2\pi t, 1 + \sin 2\pi t)$.

   (a) Show that $\int_{C_0} F = 2\pi$.

   (b) Show that $\int_{C_1} F = 0$.

9. (*) Let $F(x,y,z) = (z^3 + 2xy, x^2 + 1, 3xz^2)$ be a vector field on $\mathbb{R}^3$. Show that F is conservative by computing the partial derivatives of its component functions, and find $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $F = \nabla f$.

10. (*) Evaluate $\int_C 2xyz\, dx + x^2 z\, dy + x^2 y\, dz$, where C is a piece-wise smooth oriented curve from $(1,1,1)$ to $(1,2,4)$.