MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 1
Due: Friday, February 11, 2005

1. Read Sections 1.7–1.8 from Edwards, and re-read the Appendix.

2. (A) Prove the following result from class (2/2):

   Lemma. Let $\{\mathbf{a}_n\}$ be a convergent sequence of vectors in $\mathbb{R}^n$ with $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$. Show that $\{\mathbf{a}_n\}$ is bounded, that is, that there exists an $M > 0$ such that $||\mathbf{a}_n|| < M, \forall n \in \mathbb{N}$.

3. (*) Metric spaces.

   (a) Let S be any set, and define $d(x,x) = 0, \forall x \in S$ and $d(x,y) = 1$ if $x \neq y$. Show that (S,d) is a metric space. For the record, this particular metric is known as the *discrete metric*.

   (b) Suppose S is a metric space with distance function d. Show that S is also a metric space with new distance function given by:

   $$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

4. (*) Consider $S = \mathbb{R}^2$, with $\mathbf{v} = (a,b)$ and $\mathbf{w} = (c,e)$. We define the *Memphis metric* by $d(\mathbf{v},\mathbf{v}) = 0$ for any $\mathbf{v}$, and for $\mathbf{v} \neq \mathbf{w}$ by:

   $$d(\mathbf{v},\mathbf{w}) = \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2}$$

   (a) Show that (S,d) is a metric space.

   (b) Find $B_\varepsilon(\mathbf{0})$.

   (c) For $\mathbf{v} \neq \mathbf{0}$, find $B_\varepsilon(\mathbf{v})$, for various $\varepsilon > 0$.

5. (B) Define $f : [0,1] \longrightarrow \mathbb{R}$ as follows (note that $f(0) = 1$):

   $$f(x) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ \frac{1}{q} & , \text{ if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

   (a) Graph f.
   (Hint: Use a large scale, and plot points in a "natural" order.)

   (b) Show that f is not continuous at any rational x.

   (c) Show that f is continuous at any irrational x.

6. (C) Finish the proof of the following Theorem from class (2/4):

   Theorem. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by

   $$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

   is continuous at $\mathbf{a} = (a_1 \ldots, a_n) \in \mathbb{R}^n$ if and only if its coordinate functions $f_1, \ldots, f_m$ are continuous at $\mathbf{a}$.

7. (C) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous at $\mathbf{a}$, and suppose $g(\mathbf{a}) \neq 0$. Show that $\dfrac{f}{g}$ is continuous at $\mathbf{a}$.

8. (*) Given a subset $S \subset \mathbb{R}^n$, we define a point $x \in S$ to be an **interior point** if $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. We define the **interior** of S to be the set of interior points, and we denote it by S°. Show that the interior of any set is open.

9. (D) Let $V = \mathbb{R}^2$, and consider the following subsets:

   $$A = \mathbb{Q} \times \mathbb{Q} = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\}$$
   $$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

   For the following, recall that $\overline{S}$, S°, and S^c denote the closure, interior (see #7 for the definition), and complement, respectively, of S.

   (a) Find A°, $(A^c)^\circ$, $\overline{A}$, $\overline{A^c}$, and $\overline{A} \cap \overline{A^c}$.
   (b) Find B°, $(B^c)^\circ$, $\overline{B}$, $\overline{B^c}$, and $\overline{B} \cap \overline{B^c}$.
   (c) Find $\overline{A \cap B}$ and $(A \cap B)^\circ$.

10. (*) Let $S = \{(x, \sin(\frac{1}{x})) \mid x > 0\} \subset \mathbb{R}^2$. Find $\overline{S}$.

11. (deferred)

    A subset $S \subset \mathbb{R}^n$ is called **discrete** if, for every $x \in S$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \{x\}$, that is, the only intersection between the ball and the set is the point itself.

    (a) Show that every $f : S \longrightarrow \mathbb{R}$ is continuous if S is discrete.
    (b) Show that every closed, bounded, and discrete set is finite, and give examples why each of these three conditions is necessary.
    (c) Show that $\mathbb{Z} \subset \mathbb{R}$ is discrete.

12. (E) Let $\{S_n\}$ be a collection of open sets in $\mathbb{R}^n$, and let $\{T_n\}$ be a collection of closed sets. Show that:

    (a) $S_1 \cap S_2$ is open.
    (b) $\bigcup S_n$ is open.
    (c) $T_1 \cup T_2$ is closed.
    (d) $\bigcap T_n$ is closed.