MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 2
Due: February 18, 2005

1. Read Sections 1.7–1.8 from Edwards.

2. (E) For a set $A \subset \mathbb{R}^n$, show that $\partial A = \overline{A} \setminus A^\circ$.

3. (*) For a set $A \subset \mathbb{R}^n$, show that $\partial A = \overline{A} \cap \overline{A^c}$.

4. (*) Let $S = \{(x, \sin(\frac{1}{x})) \mid x > 0\} \subset \mathbb{R}^2$. Find $\overline{S}$.

5. (A) A subset $S \subset \mathbb{R}^n$ is called **discrete** if, for every $x \in S$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \{x\}$, that is, the only intersection between the ball and the set is the point itself.

   (a) Show that $\mathbb{Z} \subset \mathbb{R}$ is discrete.

   (b) Show that every $f : S \longrightarrow \mathbb{R}$ is continuous if S is discrete.

   (c) It is true that every closed, bounded, and discrete set is finite (see the Bolzano-Weierstrass Theorem in class). Give examples illustrating why each of these conditions is necessary.

6. (B) Let $A \subset \mathbb{R}^n$ be compact, and suppose that $B \subset A$ is closed. Use the "open cover" definition to show that B is compact.

7. (C) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous, and let $A \subset \mathbb{R}^n$ be compact. Use the "open cover" definition to show that $f(A)$ is compact.

8. (D) The Cantor Intersection Theorem states that if $\{Q_n\}_{n=1}^\infty$ is a nested (so that $Q_{n+1} \subset Q_n, \forall n$) collection of non-empty, bounded, closed sets in $\mathbb{R}^n$, then $S = \bigcap_{n=1}^{\infty} Q_n$ is also non-empty, bounded, and closed. Illustrate that the hypotheses are necessary by giving examples of the following cases:

   (a) the Q_n are nested, non-empty, and bounded, but not closed, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.

   (b) the Q_n are nested, non-empty, and closed, but not bounded, and $\bigcap_{n=1}^{\infty} Q_n = \emptyset$.