MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 3
Due: February 25, 2005

Homework Assignment #3 (Final Version)

1. Read Edwards, Sections 1.8 and 2.1. Supplementary: Fitzpatrick, Chapters 10 and 11.

2. (*) In the proof of the Lindelof Covering Theorem, we needed the following fact:

   Fact: If $A \subset \mathbb{R}^n$ is an open set and $\mathbf{x} \in A$, then there is some ball $B_\varepsilon(\mathbf{y}) \subset A$ with rational center (not necessarily $\mathbf{x}$!) and rational radius ε such that $\mathbf{x} \in B_\varepsilon(\mathbf{y})$.

For the remainder of the homework set, consider $V = \mathbb{R}^n$. Recall that we define the collection of linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \longrightarrow V \mid A \text{ is linear}\},$$

and that we may assume that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ has the structure of Euclidean space. We also define the collection of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}.$$

We further define two special subgroups of the general linear group, called the *special linear group* and the *orthogonal group*, respectively, as follows:

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid det(A) = 1\}$$

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^t A = I\}$$

(Recall that A^t is the transpose of the matrix A.)

3. (*) Show that $det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous.
   *(Hint #1: Use the theorems, not the definition.
   Hint #2: This part is important for the rest of the questions.)*

4. (D) Show that $GL_n(\mathbb{R})$ is open as a subset of $M_n(\mathbb{R})$.
   (Hint: Use #3.)

5. (D) Show that $SL_n(\mathbb{R})$ is closed (for $n \geq 1$) but not compact (for $n \geq 2$).

6. (B) Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$ by showing that:

   (a) $O_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

   (b) $O_n(\mathbb{R})$ is bounded in $M_n(\mathbb{R})$. *(Hint: What do you know about the columns of an orthogonal matrix?)*

7. (A) Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.

8. (E) Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$.
   Is X open, closed, or neither? Explain.