

MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 4
Due: March 4, 2005

Homework Assignment #4 (Final Version—significant alterations)

1. Read Edwards, Section 2.1–2.3.
2. (*) Find the equation of the line in \mathbb{R}^4 tangent to the graph of the function

$$f(t) = (a_0 + a_1 \cos t, b_0 + b_1 \sin t, t^2)$$

at a general point.

3. (B)
 - (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $\mathbf{p} = f(s_0)$ and $\mathbf{q} = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $\mathbf{p} - \mathbf{q}$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.
(Hint: The point (s_0, t_0) must be a critical point for the function $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\rho(s, t) = |f(s) - g(t)|^2$.)
 - (b) Apply the result of part (a) to find the closest pair of points of the “skew” straight lines in \mathbb{R}^3 defined by $f(s) = (s, 2s, -s)$ and $g(t) = (t + 1, t - 2, 2t + 3)$.
4. (C) Consider a particle which moves on a circular helix in \mathbb{R}^3 with position vector given by (all scalars non-zero):

$$\varphi(t) = (a \cos \omega t, a \sin \omega t, b\omega t).$$

- (a) (*) Graph $Im(\varphi)$ using Mathematica!
- (b) Show that the speed of the particle is constant.
- (c) Show that the velocity vector makes a constant nonzero angle with the z -axis.
- (d) If $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$, notice that $\varphi(t_1) = (a, 0, 0)$ and $\varphi(t_2) = (a, 0, 2\pi b)$, so the vector $\varphi(t_2) - \varphi(t_1)$ is vertical. Conclude that the equation

$$\varphi(t_2) - \varphi(t_1) = (t_2 - t_1)\varphi'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

- (e) (*) Review: Look up the Mean Value Theorem!

5. (*) Prove the “Dot Product Rule,” which says that if $f : \mathbb{R} \rightarrow \mathbb{R}^m$ and $g : \mathbb{R} \rightarrow \mathbb{R}^m$ are differentiable, then the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\varphi(a) = f(a) \cdot g(a)$ is differentiable, and $\varphi'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$, where in all instances, the products are necessarily the dot product.
6. (deferred) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ unless $x = y = 0$, in which case $f(0, 0) = 0$.
- (*) Graph f using Mathematica! (*The picture of this function was distributed in class, but it will be useful for you to be able to generate it yourself.*)
 - Show that $D_{\mathbf{v}}f(0, 0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation. (*Hint: You should conclude that $D_{\mathbf{v}}f(0, 0) = f(\mathbf{v})$.*)
 - Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.
 - Show that any differentiable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall t \in \mathbb{R}, \forall \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation $g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ and hence must be *linear*.
 - Conclude that f possesses directional derivatives in all directions at $(0, 0)$, but that f is *not* differentiable at $(0, 0)$.
7. (deferred) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0, y) = y^2$.
- Show that f is continuous at $(0, 0)$.
 - Find the partial derivatives of f at $(0, 0)$.
 - Show that f is differentiable at $(0, 0)$.
 - Show that $D_1 f$ is *not* continuous at $(0, 0)$.
8. (deferred) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$f(x, y) = (\sin(x - y), \cos(x + y)).$$

Find the equations of the tangent plane in \mathbb{R}^4 to the graph of f at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0, 0)$.

9. (deferred) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Prove that if f is differentiable at \mathbf{a} , then f is continuous at \mathbf{a} .