MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 5
Due: March 11, 2005

Homework Assignment #5 (Final Version):

1. Read Edwards, Sections 2.3–2.5.
   Next week, we will also be discussing some of Chapter 3.

2. (A) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = \frac{x^2 y}{x^2 + y^2}$ unless $x = y = 0$, in which case $f(0,0) = 0$.

   (a) (*) Graph f using Mathematica! *(The picture of this function was distributed in class, but it will be useful for you to be able to generate it yourself.)*

   (b) Show that $D_{\mathbf{v}} f(0,0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation. *(Hint: You should conclude that $D_{\mathbf{v}} f(0,0) = f(\mathbf{v})$.)*

   (c) Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.

   (d) Show that any *differentiable* function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall t \in \mathbb{R}, \forall \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation $g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$ and hence must be *linear*.

   (e) Conclude that f possesses directional derivatives in all directions at $(0,0)$, but that f is *not* differentiable at $(0,0)$.

3. (B) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2 \sin(1/x) + y^2$ for $x \neq 0$ and $f(0,y) = y^2$.

   (a) Show that f is continuous at $(0,0)$.

   (b) Find the partial derivatives of f at $(0,0)$.

   (c) Show that f is differentiable at $(0,0)$.

   (d) Show that $D_1 f$ is *not* continuous at $(0,0)$.

4. (C) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$f(x,y) = (\sin(x-y), \cos(x+y)).$$

   Find the equations of the tangent plane in $\mathbb{R}^4$ to the graph of f at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0, 0)$.

5. (C) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Prove that if f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

6. (D) Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $f(\mathbf{x}) = ||\mathbf{x}||\mathbf{x}$. Determine whether or not f is differentiable at $\mathbf{0}$. If not, why not? If so, find the first-order partial derivatives of f at $\mathbf{0}$. (Do the second-order partial derivatives of f exist at $\mathbf{0}$?)

7. (D) It may have occurred to you to ask whether a function could be differentiable at a point but not on any open set containing that point. Consider the function $f_3 : \mathbb{R} \to \mathbb{R}$ given by:

$$f_3(x) = \begin{cases} 0 & , \quad \text{if } x \notin \mathbb{Q} \\ \frac{1}{q^3} & , \quad \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

(a) Show that f_3 is not continuous at any $x \in \mathbb{Q}$.

(b) Show that f_3 is differentiable at $x = \sqrt{2}$.

8. (*) More on gradients:

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Show that:

(a) $\nabla(f + g) = \nabla f + \nabla g$.

(b) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.

(c) $\nabla(f^m) = mf^{m-1}\nabla f$, for any positive integer m.

(d) Determine a formula for $\nabla(\frac{f}{g})$ when $g(\mathbf{x}) \neq 0$.

9. (deferred) Recall the isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(a) Consider the determinant map $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, and find $\nabla(\det)(A)$, expressed in terms of $A = [a_{ij}]$.

(b) Consider the function $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.

10. (E) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x, y) = x \cdot y \cdot \dfrac{x^2 - y^2}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

(a) (*) Graph $z = f(x, y)$ using Mathematica.

(b) Show that $D_1 f(0, y) = -y$ and $D_2 f(x, 0) = x$ for all $x, y \in \mathbb{R}$.

(c) Show that $D_2 D_1 f(0, 0)$ and $D_1 D_2 f(0, 0)$ exist but are not equal.

11. (deferred) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable and has an inverse function $f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is also differentiable. Show that:

$$[J(f^{-1})](\mathbf{a}) = [(Jf)(f^{-1}(\mathbf{a}))]^{-1}.$$

Sorry for all the parentheses, but I am trying to make this clear. On the left-hand side, we are taking the Jacobian of f^{-1} and evaluating at $\mathbf{a}$. On the right-hand side, we are taking the Jacobian of f and evaluating at $f^{-1}(\mathbf{a})$, and then taking the inverse (as a matrix) of that.

12. (*) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, our theorem about cross-partials applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, the usual gradient of f, we make the following definitions:

- $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f.

- $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the *Laplacian* of f.

Finally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r \cos \theta, r \sin \theta)$.

(a) Show that $||\nabla f||^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.

(b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$.

13. (*) Following up on problem #12, if $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

With f as above, we say that f is *harmonic* on the open set $U \subset \mathbb{R}^n$ provided that $\nabla^2 f(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in U$.

(a) Find a (simple) condition on the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.

(b) Show that $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{x}) = \frac{1}{||\mathbf{x}||^{n-2}}$ is harmonic on $U = \mathbb{R}^n \setminus \{\mathbf{0}\}$.

(c) Show that if $g : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, then $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.