MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 6
Due: December 3, 2004

1. Read Chapter 4 (especially 4.1–4.3) and 6.3–6.4 of Schneider and Barker.

2. (*) Read Chapter 2 of Schneider and Barker.

3. (A) Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the linear map $L : V \longrightarrow V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$.

   (a) Write the matrix A for L with respect to the standard basis.

   (b) Find the eigenvalues of L, and find an eigenbasis for V. (Hint: Look for likely choices of eigenvalues–I claim that two of them are easy, and the third follows a pattern.)

   (c) Write the matrix B for L with respect to the eigenbasis.

   (d) Find the change of basis matrix $S : V \to V$ that takes the standard basis to the eigenbasis.

   (e) Show directly (that is, using a computation involving S) that A and B are similar.

4. (B) Let $V = \mathbb{R}^n$, and let $\mathbf{u}, \mathbf{v} \in V$. If $A : V \longrightarrow V$ is (the matrix for) a linear transformation, then define the following bilinear form:

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

   (a) Show that f_A is indeed a bilinear form.

   (b) Give a necessary and sufficient condition on the matrix A that makes f_A alternating.

   (Hint #1: You might consider the $n = 2$ case first to get a feel for this bilinear form. Hint #2: Your answers just might involve the transpose!)

5. (B) Show that not every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating by constructing an example. (Note that the only cases where this can happen are over fields F wherein $1 + 1 = 0$. Follow Halmos' reasoning in the proof that alternating implies skew-symmetric, Section 30, Theorem 1.)

6. (C) Let $dim(V) = n$, and let $f : V^k \longrightarrow F$ be a non-trivial alternating k-linear form with $k < n$. Show by example that it is possible to have a set of k linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in V such that $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$. (Make sure that $k \geq 2$ so that f can be alternating!)

7. (C) Let V be a vector space over F. Consider the set of k-linear forms $f : V^k \longrightarrow F$. For any two such forms f_1 and f_2 and any scalar $c \in F$, we define:

$$(f_1+f_2)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = f_1(\mathbf{v}_1, \ldots, \mathbf{v}_k)+f_2(\mathbf{v}_1, \ldots, \mathbf{v}_k), \qquad \forall \mathbf{v}_1, \ldots, \mathbf{v}_k \in V$$

$$(cf_1)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = c \cdot f_1(\mathbf{v}_1, \ldots, \mathbf{v}_k), \qquad \forall \mathbf{v}_1, \ldots, \mathbf{v}_k \in V$$

   (a) (*) Convince yourself that the collection of k-linear forms on V form a vector space over F with addition and scalar multiplication defined as above.

   (b) Let $V = \mathbb{R}^2$. Show that the form $f : V^2 \longrightarrow \mathbb{R}$ defined by $f((a, b), (c, d)) = ad - bc$ is bilinear and alternating.

   (c) Now let $V = \mathbb{R}^3$. Construct two *linearly independent* alternating bilinear forms $f : V^2 \longrightarrow \mathbb{R}$.

   (d) Determine the dimensions of the spaces of alternating bilinear forms on $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$.

8. (D) Recall the definition of the *transpose* of a matrix, as referred to in homework problem #5.8, and prove the following:

   Theorem: If A is an $n \times n$ matrix, then $det(A^t) = det(A)$.

9. (D) Show that $A : V \longrightarrow V$ is invertible if and only if $det(A) \neq 0$. (Note that we have used this fact several times already. The point of this exercise is to make you think carefully about the steps we used when we made the transition from alternating forms to determinants.)

10. (E) Let $A : V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space, and by slight abuse of notation, let A also be the matrix for this transformation with respect to a fixed basis. Using the following method, we determine the eigenvalues of A:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } A &\iff V_\lambda = Ker(A - \lambda I) \text{ is non-trivial} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff det(A - \lambda I) = 0 \end{aligned}$$

Thus we are inspired to make the following definition:

$p_A(\lambda) = det(A - \lambda I)$ is called the **characteristic polynomial** of A

The eigenvalues of A will be the roots of the characteristic polynomial.

(a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for A unless it is a root of $p_A(\lambda)$.

(b) If $p_A(\lambda) = (\lambda - \lambda_0)^k \cdot q(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue λ_0 has *algebraic multiplicity* equal to k. (That is, λ_0 is a root of $p_A(\lambda)$ of order k.) Show that the geometric multiplicity (which, by definition, is the dimension of the corresponding eigenspace) of an eigenvalue is less than or equal to its algebraic multiplicity.

(c) Use this method to find all eigenvalues of the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Is the matrix A diagonalizable? Explain.

(d) Repeat part (c) for the same matrix A, but considered to have complex entries.