MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 6
Due: March 18, 2005

Homework Assignment #6 (Final Version)

1. Read Edwards, Sections 2.3–2.5.
   Next week, we will also be discussing some of Chapter 3.

2. (D) Consider the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by $f(\mathbf{x}) = ||\mathbf{x}||\mathbf{x}$ from homework problem #5.6. Do the second-order partial derivatives of f exist at $\mathbf{0}$? Explain.

3. (E) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous second-order partial derivatives (so that, in particular, our theorem about cross-partials applies, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$).

   With $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, the usual gradient of f, we make the following definitions:

   - $||\nabla f||^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$ is the norm (squared) of the gradient of f.
   - $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is the *Laplacian* of f.

   Finally, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $g(r, \theta) = f(r\cos\theta, r\sin\theta)$.

   (a) Show that $||\nabla f||^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$.

   (b) Show that $\nabla^2 f = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r}$.

4. (A) Following up on problem #2, if $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partials, the *Laplacian* of f is defined to be $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$.

   With f as above, we say that f is *harmonic* on the open set $U \subset \mathbb{R}^n$ provided that $\nabla^2 f(\mathbf{x}) = 0, \ \forall \mathbf{x} \in U$.

   (a) Find a (simple) condition on the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$ that makes f harmonic.

   (b) Show that $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{x}) = \frac{1}{||\mathbf{x}||^{n-2}}$ is harmonic on $U = \mathbb{R}^n \setminus \{\mathbf{0}\}$.

   (c) Show that if $g : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, then $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = g(e^x \cos y, e^x \sin y)$ is also harmonic.

5. (B) Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable and has an inverse function $f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which is also differentiable. Show that:

$$[J(f^{-1})](\mathbf{a}) = [(Jf)(f^{-1}(\mathbf{a}))]^{-1}.$$

Sorry for all the parentheses, but I am trying to make this clear. On the left-hand side, we are taking the Jacobian of f^{-1} and evaluating at $\mathbf{a}$. On the right-hand side, we are taking the Jacobian of f and evaluating at $f^{-1}(\mathbf{a})$, and then taking the inverse (as a matrix) of that.

6. (C) Recall that we have an isomorphism of vector spaces $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

   (a) Consider the determinant map $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$, and find $\nabla(\det)(A)$, expressed in terms of $A = [a_{ij}]$.

   (b) Consider the function $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ given by $f(A) = A^2$. Show that $Jf_A(H) = AH + HA$.