MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 7
Due: December 10, 2004

1. Read the Appendix to Edwards (especially Theorems A.1 and A.6–A.8), Section 1.3 of Edwards, and Chapter 7 (especially sections 7.1–7.3) of Schneider and Barker.

2. (*) Write down a real number that: 1. you have never seen written down before, and 2. does not have a finite algebraic representation in terms of standard mathematical symbols. (For example, $\pi + 1$ is certainly a real number, but it does not satisfy condition 2.)

3. (*) Prove that $\sqrt[3]{2}$ is an irrational real number. (In other words, show that there is a real number x that satisfies the equation $x^3 - 2 = 0$, but that x is not rational.)

4. (*) Prove that $\sqrt{5}$ is an irrational real number. (Hint: Think about equivalence classes in $\mathbb{Z}/5\mathbb{Z}$, or about the remainders when using the Division Algorithm in $\mathbb{Z}$.)

5. (*) Prove that the sum of a rational number and an irrational real number must be an irrational real number.

6. (A) Let a and b be real numbers. Using P to denote the set of positive elements, we define "less than" formally by the statement:

$$a < b \quad \text{iff} \quad b - a \in P.$$

   Use axioms P1–P3 for an ordered field and the definition of absolute value to show that:

   - If $0 < a < b$, then $0 < a^2 < b^2$.
   - If $a^2 < b^2$, then $|a| < |b|$.

7. (B) Use the definition of Cauchy sequence to show that the sequence of rational numbers $\left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty}$ is a Cauchy sequence.

8. (C) Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, name the equivalence class that acts as the multiplicative identity, and verify that it does. Note that this includes showing that the proposed multiplicative identity is, in fact, an equivalence class of Cauchy sequences!

9. (C) Considering the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers, prove the existence of multiplicative inverses (for elements other than the additive identity). Note that this includes showing that a proposed multiplicative inverse is, in fact, an equivalence class of Cauchy sequences!

10. (*) Use the Completeness Axiom to prove that $\mathbb{N}$ is not bounded above as a set of real numbers.

11. (D) In this problem, we will show that the *golden ratio*, $\varphi = \frac{1+\sqrt{5}}{2}$, is a real number because it is the supremum of a non-empty bounded set of real numbers. (More precisely, we will show that φ is the limit of a bounded, increasing sequence.)

    Consider the recursively defined sequence:

    $$a_1 = 1 \quad \text{and} \quad a_{n+1} = \sqrt{1 + a_n}, \quad \text{for } n \geq 1$$

    (a) Use induction to show that $a_n \leq 2$ for all $n \in \mathbb{N}$.

    (b) Use induction to show that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

    (c) Since $L = \lim_{n\to\infty} a_n$ exists by the completeness axiom for $\mathbb{R}$, show that L satisfies the equation $L^2 - L - 1 = 0$ by considering the expression $\lim_{n\to\infty} a_{n+1}^2 - a_n - 1$.

    (Hint: For this part, you may use theorems about limits such as the fact that $\lim_{n\to\infty} (a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$ provided that these limits exist.)

    (d) Show that $L = \varphi$.

12. (E) In this problem, you will show that every real number (with a few exceptions!) has a unique decimal expansion.

    Let $x > 0$ be a real number. Show that there is an integer k and integers $a_i \in \{0, 1, 2, \ldots, 9\}$ for every $i \geq k$ such that x may be represented in the form:

    $$\begin{aligned} x &= \sum_{i=k}^{\infty} a_i \cdot 10^{-i} \\ &= a_k \cdot 10^{-k} + a_{k+1} \cdot 10^{-k-1} + \cdots + a_{-1} \cdot 10^1 + a_0 \cdot 10^0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots \end{aligned}$$

    Show that this representation is unique, except in the case where there exists some $n \in \mathbb{N}$ such that $10^n \cdot x \in \mathbb{N}$.

    (Hints: Use the Well-Ordering Principle of the Natural Numbers to establish that there is a "first term," that is, some k at which the series begins, and then perhaps the Division Algorithm and the greatest integer function to find the coefficients.)