MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 7
Due: April 8, 2005

Homework Assignment #7 (Final Version)

1. Read the proof of the Inverse Function Theorem.
2. Read Edwards, Chapter 3 and Sections 2.6–2.8.
3. (A) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^2 + y^3 + e^y$, and let $C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ be a level set of f.
   (a) Show that for every point of C, there is a neighborhood of the point in which y may be defined implicitly as a function of x.
   (b) Find $\frac{dy}{dx}$ at every point of C.
4. (*) Show that the equations $\sin(x + z) + \log yz^2 = 0$ and $e^{x+z} + yz = 0$ define z implicitly as a function of x and y near the point $(1, 1, -1)$.
5. (A) Consider the set S of points in $\mathbb{R}^5$ defined by the two equations:
$$xu^2 + yzv + x^2z = 3$$
$$xyv^3 + 2zu - u^2v^2 = 2$$
   Show that there is a neighborhood of the point $(1, 1, 1, 1, 1) \in S$ and a differentiable function $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that in the neighborhood, the point $(x, y, z, u, v) \in S$ where $h(x, y, z) = (u, v)$, and find $Jh(1, 1, 1)$.
6. (B) In the following problem, we consider the notion of the *local invertibility* of a function and the relationship between this condition and that of injectivity.
   (a) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally invertible at every point of its domain (that is, $f'(a) \neq 0, \forall a \in \mathbb{R}$). Show that f is one-to-one.
   (b) Consider $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $g(x, y) = (e^x \cos y, e^x \sin y)$. Show that g is locally invertible at every point of its domain (that is, $\det[Jg(\mathbf{x})] \neq 0, \forall \mathbf{x} \in \mathbb{R}^2$), but that g is not one-to-one.
7. (B) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Show that f is locally invertible in a neighborhood of every point except the origin, and compute f^{-1} explicitly.

8. (C) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

(a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

(b) Show that there is no open set (in this case, an open interval) containing 0 on which f is one-to-one.

9. (D) Suppose $\varphi : \mathbb{R}^3 \to \mathbb{R}$ and $\psi : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by:

$$F(\mathbf{x}) = (\varphi(\mathbf{x}), \psi(\mathbf{x}), 1 + \varphi(\mathbf{x})\psi(\mathbf{x}) + \varphi(\mathbf{x})^3).$$

(a) Explain analytically (using the Jacobian) why there is no point $\mathbf{x} \in \mathbb{R}^3$ at which the assumptions of the Inverse Function Theorem hold for F.

(b) Explain geometrically (using the topology of Euclidean space) why there is no point $\mathbf{x} \in \mathbb{R}^3$ at which the conclusion of the Inverse Function Theorem holds for F.