MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 8
Due: MONDAY, December 20, 2004

1. Read Chapter 7 (especially sections 7.1–7.3) of Schneider and Barker and Sections 1.3 and 1.6 of Edwards.

2. (A) Recall homework problem #6.4 in which you showed that for $V = \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} \in V$, if $A : V \longrightarrow V$ is a linear transformation, then

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

defines a bilinear form.

Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

3. (B) Consider the real vector space $V = C[0, 1]$ of continuous real-valued functions defined on the closed interval $[0, 1]$, and define the bilinear form $\langle \cdot\, , \, \cdot \rangle : V^2 \to \mathbb{R}$ by:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$$

In class, we checked that this form is indeed bilinear, and it is clear that it is symmetric and positive. Use facts from single-variable Calculus to prove that this form is positive-definite.

(Hint: Look up the δ-ε definition of continuity!)

4. (C) Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $|| \cdot ||$. We say that a linear transformation $A : V \longrightarrow V$ is *norm-preserving* if $||A\mathbf{v}|| = ||\mathbf{v}||$ for every $\mathbf{v} \in V$ and *inner-product-preserving* if $\langle A\mathbf{u}, A\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

   (a) Show that A is inner-product-preserving if and only if it is norm preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $||A(\mathbf{u} + \mathbf{v})||^2 = ||\mathbf{u} + \mathbf{v}||^2$.)

   (b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

5. (D) A vector space V is called a *normed linear space* if there exists a map $||\cdot|| : V \to \mathbb{R}$ called the **norm** satisfying:

   i. $||c \cdot \mathbf{v}|| = |c| \cdot ||\mathbf{v}||, \forall c \in \mathbb{R}, \forall \mathbf{v} \in V$
   ii. $||\mathbf{v}|| \geq 0, \forall \mathbf{v} \in V$, and $||\mathbf{v}|| = 0$ iff $\mathbf{v} = \mathbf{0}$.
   iii. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||, \forall \mathbf{u}, \mathbf{v} \in V$

   In Euclidean space, the map $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is the norm *associated* to the inner product.

   Show that the function $||\cdot|| : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

   $$||(x_1, \ldots, x_n)|| = \max\{|x_1|, \ldots, |x_n|\}$$

   defines a norm on $\mathbb{R}^n$. When $n > 1$, show that there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that this norm is associated to the inner product.

6. (E) Orthogonalizing a set of functions:

   (a) Consider the vector space $V = C[-1, 1]$ of real-valued continuous functions on the closed interval $[-1, 1]$ with inner product

   $$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

   Orthogonalize the set of functions $\{1, x, x^2, x^3\}$ with respect to this inner product.

   (b) With $V = C[0, 1]$ and inner product

   $$\langle f(x), g(x) \rangle = \int_{0}^{1} f(x)g(x)\, dx,$$

   orthogonalize the same set of functions from part (a).

   (c) Now let $V = C[-1, 1]$ as in part (a), but define $\langle \cdot, \cdot \rangle$ as in part (b). Show that this bilinear form is *not* an inner product on V. (Which properties of an inner product *are* satisfied?)

7. (*) Let V be a real inner product space, and let $\mathbf{u} \in V$ be a fixed non-zero vector. Let $P_{\mathbf{u}} : V \to V$ be the orthogonal projection in the direction of $\mathbf{u}$ given, as in class, by:

   $$P_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$

   Show that $P_{\mathbf{u}} \circ P_{\mathbf{u}} = P_{\mathbf{u}}$.